Turnpike phenomenon for symmetric variational problems

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Assume that $f : R^n \times R^n \to R^1$ is a bounded from below borelian function such that

$$f(x, y) = f(x, -y) \text{ for all } x, y \in R^n,$$

there exists $(\bar{x}, \bar{y}) \in R^n \times R^n$ such that

$$f(\bar{x}, \bar{y}) = \text{inf}(f) := \text{inf}\{f(\xi, \eta) : \ \xi, \eta \in R^n\},$$

$$\{(x, y) \in R^n \times R^n :$$

$$f(x, y) = \text{inf}(f)\} = \{(\bar{x}, \bar{y}), \ (\bar{x}, -\bar{y})\}$$

and that the following assumptions hold:

(A1) for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $(x, y) \in R^n \times R^n$ satisfying

$$f(x, y) \leq \inf(f) + \delta$$

the inequalities

$$|x - \bar{x}| \leq \epsilon$$

and

$$\min\{|y - \bar{y}|,\ |y + \bar{y}|\} \leq \epsilon$$

hold;

(A2) for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $(x, y) \in R^n \times R^n$ satisfying

$$|x - \bar{x}| \leq \delta,\ |y - \bar{y}| \leq \delta$$

the inequality

$$f(x, y) \leq f(\bar{x}, \bar{y}) + \epsilon$$

is true.

Assumption (A2) means that the function f is continuous at the point $(\bar{x}, \bar{y})$ while assumption (A1) means that the minimization problem

$$f(x, y) \to \text{min},\ x, y \in R^n$$

is well posed. According to the results of Chapter 2 of Alexander J. Zaslavski, Turnpike Phenomenon and Symmetric Optimization Problems, Springer, 2022 this well-posedness property holds for most symmetric objective functions.

Let $a > 0$ and $\psi : [0, \infty) \to [0, \infty)$ be an increasing function satisfying

$$\lim_{t \to \infty} \psi(t) = \infty.$$

Assume that the following assumption holds:

(A3) the function f is bounded on all bounded sets and for each $(x, u) \in R^{\times} R^n$,

$$f(x, u) \geq \psi(|u|)|u| - a.$$

For each pair of nonnegative numbers $T_1 < T_2$ and each $y, z \in R^n$ we denote by $W^{1,1}(T_1, T_2)$ the set of all a. c. functions $x : [T_1, T_2] : R^n$, consider the problems

$$\int_{T_1}^{T_2} f(x(t), x'(t))dt \to \min, \qquad (P_{T_1,T_2})$$

$$x \in W^{1,1}(T_1, T_2),$$

$$\int_{T_1}^{T_2} f(x(t), x'(t))dt \to \min, \qquad (P_{T_1,T_2,y})$$

$$x \in W^{1,1}(T_1, T_2),\ x(T_1) = y,$$

$$\int_{T_1}^{T_2} f(x(t), x'(t))dt \to \min, \qquad (P_{T_1,T_2,y,z})$$

$$x \in W^{1,1}(T_1, T_2),\ x(T_1) = y,\ x(T_2) = z$$

and define

$$U(T_1,T_2) = \inf\{\int_{T_1}^{T_2} f(x(t),x'(t))dt :$$

$$x \in W^{1,1}(T_1,T_2)\},$$

$$U(T_1,T_2,y) = \inf\{\int_{T_1}^{T_2} f(x(t),x'(t))dt :$$

$$x \in W^{1,1}(T_1,T_2),\ x(T_1) = y\},$$

$$U(T_1,T_2,y,z) = \inf\{\int_{T_1}^{T_2} f(x(t),x'(t))dt :$$

$$x \in W^{1,1}(T_1,T_2),\ x(T_1) = y,\ x(T_2) = z\}.$$

There are two cases: $\bar{y} = 0$; $\bar{y} \neq 0$. If $\bar{y} = 0$, then for each $T_2 > T_1 \geq 0$, the function $x(t) = \bar{x}$, $t \in [T_1,T_2]$ is a solution of the problems (P_{T_1,T_2}), $(P_{T_1,T_2,\bar{x}})$, $(P_{T_1,T_2,\bar{x},\bar{x}})$.

For each pair of numbers $T_1 < T_2$ and each $x \in W^{1,1}(T_1, T_2)$ set

$$I(T_1, T_2, x) = \int_{T_1}^{T_2} f(x(t), x'(t))dt.$$

In Chapter 5 of our book we prove the following result.

Theorem 1 *Let $T > 0$. Then*

$$U(0, T) = U(0, T, \bar{x}) = U(0, T, \bar{x}, \bar{x}) = Tf(\bar{x}, \bar{y}).$$

Moreover, for each $\epsilon > 0$ there exists $x \in W^{1,1}(0, T)$ such that

$$x(0) = x(T) = \bar{x},$$

$$I(0, T, x) \leq Tf(\bar{x}, \bar{y}) + \epsilon,$$

$$|x(t) - \bar{x}| \leq \epsilon, \ t \in [0, T],$$

$$x'(t) \in \{\bar{y}, \ -\bar{y}\}, \ t \in [0, T] \ \textit{a. e.}.$$

Note that if $\bar{y} \neq 0$, then for every positive T problems $(P_{0,T})$, $(P_{0,T,\bar{x}})$ and $(P_{0,T,\bar{x},\bar{x}})$ have no minimizers.

The following useful result is also obtained in Chapter 5 of AZ 2022.

Theorem 2 *Let $L_0, M_0 > 0$. Then there exist $M_1 > 0$ such that for each $T > L_0$ and each $y, z \in R^n$ satisfying $|y|, |z| \leq M_0$ the inequality*

$$U(0, T, y, z) \leq Tf(\bar{x}, \bar{y}) + M_1$$

holds.

We denote by mes(Ω) the Lebesgue measure of a Lebesgue measurable set $\Omega \subset R^1$

The next theorem is the first turnpike result of Chapter 5. It shows that for approximate solutions x of our variational problems on intervals $[0, T]$, where T is sufficiently large and given values $x(0), x(T)$ at the end points belong to a given bounded set C, the Lebesgue measure of all points $t \in [0, T]$ such that $(x(t), x'(t))$ does not belong to an ϵ-neighborhood of the set $\{(\bar{x}, \bar{y}),\ (\bar{x}, -\bar{y})\}$ does not exceed a constant L which depends only on ϵ and the set C and does not depend on $T, x(0), x(T)$. In the literature this property is known as the weak turnpike property.

Theorem 3 *Let $\epsilon \in (0,1)$ and $L_0, M_0, M_1 > 0$. Then there exists $L_1 > L_0$ such that for each $T > L_1$ and each $x \in W^{1,1}(0,T)$ such that*

$$|x(0)| \le M_0$$

and at least one of the following conditions holds:

(a)

$$|x(T)| \le M_0,\ I^f(0,T,x) \le U(0,T,x(0),x(T)) + M_1;$$

(b)

$$I^f(0,T,x) \le U(0,T,x(0)) + M_1$$

the inequality

$$\textit{mes}(\{t \in [0,T] :\ \max\{|x(t) - \bar{x}|,$$

$$\min\{|x'(t) - \bar{y}|,\ |x'(t) + \bar{y}|\}\} > \epsilon\}) \le L_1.$$

The following theorem is also proved in Chapter 5. It shows that for approximate solutions x of our variational problems on intervals $[0, T]$, where T is sufficiently large and given values $x(0), x(T)$ at the end points belong to a given bounded set C, the set of all points $t \in [0, T]$ such that $x(t)$ does not belong to an ϵ-neighborhood of $\bar{x}$ is contained in the union of two intervals, where the first interval contains 0, the second one contains T and their lengths do not exceed a constant L which depends only on ϵ and the set C and does not depend on $T, x(0), x(T)$. In the literature this property is known as the turnpike property.

Theorem 4 *Let* $\epsilon \in (0,1]$. *Then there exist* $L, \delta > 0$ *such that for each* $T > 2L$ *and each* $u \in W^{1,1}(0,T)$ *such that*

$$|u(0)| \leq M$$

and at least of the following conditions holds:

$$|u(T)| \leq M,\ I(0,T,u) \leq U(0,T,u(0),u(T)) + \delta;$$

$$I(0,T,x) \leq U(0,T,u(0)) + \delta$$

there exist $\tau_1 \in [0,L],\ \tau_2 \in [T-L,T]$ *such that*

$$|u(t) - \bar{x}| \leq \epsilon,\ t \in [\tau_1, \tau_2].$$

Moreover, if $|u(0) - \bar{x}| \leq \delta$, *then* $\tau_1 = 0$ *and if* $|u(T) - \bar{x}| \leq \delta$, *then* $\tau_2 = T$.

Chapter 5 contains many other turnpike results, including the results of the stability of the turnpike phenomenon under small perturbations of integrands and on the turnpike phenomenon in the regions close to the right end points. The results of Chapter 5 are obtained for variational problems in infinite dimensional spaces.

In Chapter 5 of our book we prove our second weak turnpike result. It shows that for approximate solutions x of our variational problems on intervals $[0, T]$, where T is sufficiently large and given values $x(0), x(T)$ at the end points belong to a given bounded set C, the set of all points $t \in [0, T]$ such that $x(t)$ does not belong to an ϵ-neighborhood of $\bar{x}$ is contained in the union of a finite family of intervals, their lengths do not exceed a constant l and their number does not exceed a constant Q which depend only on ϵ and the set C and does not depend on $T, x(0), x(T)$.

Theorem 5 *Let*

$$M_0 > \|\bar{x}\| + \|\bar{y}\| + 1,$$

$M_1 > 0$ *and* $\epsilon \in (0, 1]$. *Then there exist* $l > 0$ *and a natural number* Q *such that for each* $T > lQ$ *and each* $x \in W^{1,1}(0, T)$ *such that*

$$\|x(0)\| \leq M_0$$

and at least of the conditions below holds:

(a)

$$\|x(T)\| \leq M_0,\ I(0, T, x) \leq U(0, T, x(0), x(T)) + M_1;$$

(b)

$$I(0, T, x) \leq U(0, T, x(0)) + M_1$$

there exists a sequence of closed intervals $[a_i, b_i] \subset [0, T],\ i = 1, \ldots, q$ *such that*

$$q \leq Q,$$

$$0 \leq b_i - a_i \leq l,\ i = 1, \ldots, q,$$

$$a_{i+1} \geq b_i \text{ for all } i \in \{1, \ldots, q\} \setminus \{q\}$$

and

$$\{t \in [0, T] : \|x(t) - \bar{x}\| > \epsilon\} \subset \cup_{i=1}^{q} [a_i, b_i].$$

The space of integrands

Denote by $\mathfrak{M}$ the set of all borelian functions $g : X \times X \to R^1$ such that

$$g(x,u) \geq \psi(\|u\|)\|u\| - a$$

for each $(x,u) \in X \times X$.

For each pair of nonnegative numbers $T_1 < T_2$ each $x \in W^{1,1}(T_1,T_2)$, each $g \in \mathfrak{M}$ and each $y,z \in X$ define

$$I^g(T_1,T_2,x) = \int_{T_1}^{T_2} g(x(t),x'(t))dt,$$

$$U^g(T_1,T_2) = \inf\{I^g(T_1,T_2,x) : \ x \in W^{1,1}(T_1,T_2)\},$$

$$U^g(T_1,T_2,y) = \inf\{I^g(T_1,T_2,x) :$$

$$x \in W^{1,1}(T_1,T_2), \ x(T_1) = y\},$$

$$U^G(T_1,T_2,y,z) = \inf\{I^g(T_1,T_2,x) :$$

$$x \in W^{1,1}(T_1,T_2), \ x(T_1) = y, \ x(T_2) = z\}.$$

For each natural number n denote by $E_w(n)$ the set of all pairs $(g,h) \in \mathfrak{M} \times \mathfrak{M}$ which have the following properties:

for each $y,z \in B(0,n)$,

$$|g(y,z) - h(y,z))| \leq n^{-1};$$

for each $y,z \in X$ satisfying

$$\min\{g(y,z),\ h(y,z)\} \leq n$$

the inequality

$$|g(y,z) - h(y,z)| \leq n^{-1}$$

holds.

For the space $\mathfrak{M}$ there exists a uniformity generated by the base $E_w(n)$, $n = 1, 2, \ldots$ which induces in $\mathfrak{M}$ a topology.

We continue to consider the function f introduced before.

The following result shows that the turnpike phenomenon is stable under small perturbations of the integrand f.

Theorem 6 *Let $\epsilon > 0$ and*

$$M > \|\bar{x}\| + \|y\| + 1.$$

Then there exist $\delta, L > 0$ and a neighborhood $\mathcal{U}$ of f in $\mathfrak{M}$ such that for each $g \in \mathcal{U}$, each $T > 2L$ and each $u \in W^{1,1}(0,T)$ such that at least of the conditions below:

(a)

$$u((0),\ u(T) \in B(0, M_0),$$

$$I^g(0,T,u) \leq U^g(0,T,u(0),u(T)) + \delta;$$

(b)

$$u(0) \in B(0, M),$$

$$I^g(0, T, u) \le U^g(0, T, u(0)) + \delta$$

there exist $\tau_1 \in [0, L]$, $\tau_2 \in [T - L, T]$ *such that*

$$\|u(t) - \bar{x}\| \le \epsilon, \ t \in [\tau_1, \tau_2].$$

Moreover, if $\|u(0) - \bar{x}\| \le \delta$*, then* $\tau_1 = 0$ *and if* $\|u(T) - \bar{x}\| \le \delta$*, then* $\tau_2 = T$*.*

Theorem 7 *Let*

$$M > \|\bar{x}\| + \|\bar{y}\| + 1$$

and $\epsilon \in (0, 1]$. *Then there exist* $L > 0$, *a natural number* Q *and a neighborhood* $\mathcal{U}$ *of* f *in* $\mathfrak{M}$ *such that for each* $T > lQ$, *each* $g \in \mathcal{U}$ *and each* $u \in W^{1,1}(0, T)$ *such that*

$$\|u(0)\| \le M$$

and at least one of the conditions below holds:

(a)

$$\|u(T)\| \le M, \ I^g(0, T, u) \le U^g(0, T, u(0), u(T)) + M;$$

(b)

$$I^g(0, T, u) \le U^g(0, T, u(0)) + M$$

there exists a sequence of closed intervals $[a_i, b_i] \subset [0, T]$, $i = 0, \ldots, q$, *where* $q \ge 0$ *is an integer, such that*

$$0 \le q \le Q,$$

$$0 \le b_i - a_i \le L,\ i = 0, \dots, q,$$

$$a_{i+1} \ge b_i \text{ for all } i \in \{0, \dots, q\} \setminus \{q\}$$

and

$$\{t \in [0, T] :\ \|u(t) - \bar{x}\| > \epsilon\} \subset \cup_{i=0}^{q} [a_i, b_i].$$

Alexander J. Zaslavski, Turnpike Phenomenon and Symmetric Optimization Problems, Springer, 2022