Fractional generalization of FPUT media and a variational method

Jorge E. Macías-Díaz
jorgmd@tlu.ee

Tulane University

XXI International Conference on
Computational and Mathematical Methods in Science and Engineering
Cádiz, Spain

November 2022

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The α- and β-**Fermi–Pasta–Ulam–Tsingou (FPUT) chains** are:

- $$\frac{d^2u_n}{dt^2} = u_{n+1} - 2u_n + u_{n-1} + \alpha\left[(u_{n+1}-u_n)^2 - (u_n - u_{n-1})^2\right], \text{ for } 1 \le n \le N,$$
- $$\frac{d^2u_n}{dt^2} = u_{n+1} - 2u_n + u_{n-1} + \beta\left[(u_{n+1}-u_n)^3 - (u_n - u_{n-1})^3\right], \text{ for } 1 \le n \le N.$$

These chains exhibit many nonlinear phenomena:

1. *Chaos* and *equipartition*.
2. *Recurrence*.
3. *Supratransmission* and *bistability*.

Continuous extension

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\epsilon}{s+1}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^{s+1}.$$

Let $B = (a, b) \subseteq \mathbb{R}$ and $\Omega = B \times (0, T) \subseteq \mathbb{R}^2$. If $u : \overline{\Omega} \to \mathbb{R}$, then $u(x, t) = 0$, for each $(x, t) \in (\mathbb{R} \setminus \overline{B}) \times [0, T]$. Let $1 \leq q < \infty$, and

$$L_{x,q}(\overline{\Omega}) = \left\{ f : f(\cdot, t) \in L_q(\overline{B}) \text{ for each } t \in [0, T] \right\}.$$

Define

$$\|f\|_{x,q} = \left(\int_{\overline{B}} |f(x, t)|^q dx \right)^{1/q}, \quad \forall f \in L_{x,q}(\overline{\Omega}),$$

$$\langle f, g \rangle_x = \int_{\overline{B}} f(x, t) g(x, t) dx, \quad \forall f, g \in L_{x,2}(\overline{\Omega})$$

Definition

Let $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$ be such that $n - 1 < \alpha < n$. The **Riesz fractional derivative** of $u : \overline{\Omega} \to \mathbb{R}$ of order α with respect to x is

$$\frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} = \frac{-1}{2 \cos(\frac{\pi\alpha}{2}) \Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{\infty} \frac{u(\xi, t) d\xi}{|x - \xi|^{\alpha + 1 - n}}, \quad \forall (x, t) \in \Omega.$$

If $\alpha \in (1,2]$ and $u, v \in H_2(\overline{\Omega})$, then

$$\left\langle -\frac{\partial^\alpha u}{\partial |x|^\alpha}, v \right\rangle = \left\langle u, -\frac{\partial^\alpha v}{\partial |x|^\alpha} \right\rangle = \left\langle \frac{\partial^{\alpha/2} u}{\partial |x|^{\alpha/2}}, \frac{\partial^{\alpha/2} v}{\partial |x|^{\alpha/2}} \right\rangle, \quad \forall t \in [0, T].$$

- Let $p \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_p \in (1,2]$ be constants.
- Let $G, G_1, G_2, \ldots, G_p : \mathbb{R} \to \mathbb{R}$ be functions.

Consider the **IBVP**

$$\boxed{\begin{array}{l} \dfrac{\partial^2 u(x,t)}{\partial t^2} + \gamma \dfrac{\partial u(x,t)}{\partial t} + \dfrac{dG(u(x,t))}{dv} = \\ \qquad = \displaystyle\sum_{j=1}^{p} \frac{\partial^{\alpha_j/2}}{\partial |x|^{\alpha_j/2}} \frac{dG_j}{dv} \left(\frac{\partial^{\alpha_j/2} u(x,t)}{\partial |x|^{\alpha_j/2}} \right), \qquad \forall (x,t) \in \Omega, \\ \text{such that } \begin{cases} u(x,0) = \phi(x), & \forall x \in B, \\ \dfrac{\partial u(x,0)}{\partial t} = \psi(x), & \forall x \in B, \\ u(x,t) = 0, & \forall (x,t) \in \partial B \times [0,T]. \end{cases} \end{array}}$$

The associated **local energy density** is

$$\boxed{\mathcal{H}(x,t) = \frac{1}{2}\left(\frac{\partial u(x,t)}{\partial t}\right)^2 + G(u(x,t)) + \sum_{j=1}^{p} G_j\left(\frac{\partial_x^{\alpha_j/2} u(x,t)}{\partial |x|^{\alpha_j/2}}\right).}$$

The **total energy** at time $t \in (0,T)$ is given by $\boxed{\mathcal{E}(t) = \int_{\mathbb{R}} \mathcal{H}(x,t)dx.}$

Particular fractional models

Let $\gamma = 0$ and $p = 1$.

- **sG equation:** $G(v) = 1 - \cos(v)$ and $G_1(v) = \frac{1}{2}v^2$.
- **NLKG equation:** $G(v) = \frac{1}{2!}v^2 - \frac{1}{4!}v^4$ and $G_1(v) = \frac{1}{2}v^2$.
- **DsG equation:** $G(v) = 1 - \frac{1}{3}\cos v - \frac{1}{6}\cos(2v)$ and $G_1(v) = \frac{1}{2}v^2$.
- **FPUT equation:** $G(v) = 0$ and $G_1(v) = \frac{1}{2}v^2 + \frac{\epsilon}{s+1}v^{s+1}$, where $\epsilon > 0$ and $s \geq 2$.
- **Esastic string:** $G(v) = 0$ and $G_1(v) = \sqrt{1+v^2}$.

Theorem (Dissipation of energy)

If u satisfies **IBVP**, *then* $\mathcal{E}'(t) = -\gamma\|u_t\|_{x,2}^2$.

Proof.

Notice that

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial t}\right\|_{x,2}^2 = \left\langle \frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial t}\right\rangle_x,$$

$$\frac{d}{dt}\int_{\mathbb{R}} G(u(x,t))dx = \left\langle \frac{dG(u)}{dv}, \frac{\partial u}{\partial t}\right\rangle_x,$$

$$\frac{d}{dt}\int_{\mathbb{R}} G_j\left(\frac{\partial_x^{\alpha_j/2}u(x,t)}{\partial|x|^{\alpha_j/2}}\right)dx = -\left\langle \frac{\partial^{\alpha_j/2}}{\partial|x|^{\alpha_j/2}}\frac{dG_j}{dv}\left(\frac{\partial_x^{\alpha_j/2}u}{\partial|x|^{\alpha_j/2}}\right), \frac{\partial u}{\partial t}\right\rangle_x.$$

Take then the derivative of $\mathcal{E}(t)$ to reach the conclusion. $\square$

Corollary (Energy positivity)

Suppose that $G, G_1, G_2, \ldots, G_p$ are all nonnegative. Then

$$\mathcal{E}(t) = \frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|_{x,2}^2 + \|G(u)\|_{x,1} + \sum_{j=1}^{p}\left\|G_j\left(\frac{\partial_x^{\alpha_j/2}u}{\partial|x|^{\alpha_j/2}}\right)\right\|_{x,1}, \quad \forall t \in (0,T).$$

Boundedness is obtained as a consequence. Indeed, notice that

$$\begin{aligned}\max\left\{\left\|\frac{\partial u}{\partial t}\right\|_{x,2}^2, \|G(u)\|_{x,1}\right\} \vee \max\left\{\left\|G_j\left(\frac{\partial_x^{\alpha_j/2}u}{\partial|x|^{\alpha_j/2}}\right)\right\|_{x,1} : j \in I_p\right\} \\ \leq 2\mathcal{E}(t) = 2\mathcal{E}(0) - 2\gamma\int_0^t \|u_t\|_{x,2}^2 dt \leq 2\mathcal{E}(0) = C,\end{aligned}$$

where

$$C = \|\psi\|_{x,2}^2 + 2\|G(\phi)\|_{x,1} + 2\sum_{j=1}^{p}\left\|G_j\left(\frac{\partial_x^{\alpha_j/2}\phi}{\partial|x|^{\alpha_j/2}}\right)\right\|_{x,1}.$$

Contents

Fix $M, N \in \mathbb{N}$, and use the step-sizes $h = (b-a)/M$ and $\tau = T/N$. Fix partitions $x_m = a + mh$ and $t_n = n\tau$, for $m \in \bar{I}_M$ and $n \in \bar{I}_N$. Let

$$\mathcal{V}_h = \left\{ f : \{x_m\}_{m=0}^{M} \to \mathbb{R} : f(x_0) = f(x_M) = 0 \right\}.$$

Agree that $f_m = f(x_m)$, and let $U_m^n \approx u(x_m, t_n)$. Set

$$U^n = (U_0^n, U_1^n, \ldots, U_M^n) \in \mathcal{V}_h.$$

For each $U, V \in \mathcal{V}_h$ and $1 \leq p < \infty$, define

$$\langle U, V \rangle = h \sum_{m \in \bar{I}_M} U_m V_m,$$

$$\|U\|_p = \left[h \sum_{m \in \bar{I}_M} |U_m|^p \right]^{1/p},$$

$$\|U\|_\infty = \max\{|U_m| : m \in \bar{I}_M\}.$$

Let $(U^n) \subseteq \mathcal{V}_h$. Introduce the **difference operators**

$$\delta_t U_m^{n+\frac{1}{2}} = \frac{U_m^{n+1} - U_m^n}{\tau}, \qquad \mu_t U_m^{n+\frac{1}{2}} = \frac{U_m^{n+1} + U_m^n}{2},$$

$$\delta_t^{(2)} U_m^n = \frac{U_m^{n+1} - 2U_m^n + U_m^{n-1}}{\tau^2}, \qquad \hat{\delta}_t^{(2)} U_m^{n+\frac{1}{2}} = \delta_t^{(2)} \mu_t U_m^{n+\frac{1}{2}}.$$

Definition

Let $f : \mathbb{R} \to \mathbb{R}$, $h > 0$ and $\alpha > -1$. The **fractional centered difference** of order α of f at the point x as

$$\Delta_h^\alpha f(x) = \sum_{k=-\infty}^{\infty} g_k^{(\alpha)} f(x - kh), \quad \forall x \in \mathbb{R},$$

where

$$g_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha + 1)}{\Gamma(\frac{\alpha}{2} - k + 1)\Gamma(\frac{\alpha}{2} + k + 1)}, \quad \forall k \in \mathbb{Z}.$$

Let $0 < \alpha \leq 2$ and $\alpha \neq 1$.

1. The coefficients $(g_k^{(\alpha)})_{k=-\infty}^{\infty}$ satisfy
$$g_0^{(\alpha)} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha/2+1)^2},$$
$$g_{k+1}^{(\alpha)} = \left(1 - \frac{\alpha+1}{\alpha/2+k+1}\right) g_k^{(\alpha)}, \quad \forall k \in \mathbb{N} \cup \{0\}.$$
2. $g_0^{(\alpha)} \geq 0$.
3. $g_k^{(\alpha)} = g_{-k}^{(\alpha)} < 0$ for all $k \geq 1$, and
4. $\displaystyle\sum_{k=-\infty}^{\infty} g_k^{(\alpha)} = 0$. It follows that $g_0^{(\alpha)} = -\displaystyle\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} g_k^{(\alpha)}$.

Theorem

If $\alpha \in (0,1) \cup (1,2]$ and f has its derivatives up to order 5 in $L^1(\mathbb{R})$:

$$-\frac{1}{h^\alpha}\Delta_h^\alpha f(x) = \frac{d^\alpha f(x)}{d|x|^\alpha} + \mathcal{O}(h^2), \quad \forall x \in \mathbb{R}.$$

Let $U \in \mathcal{V}_h$ and $\alpha \in (0,1) \cup (1,2]$. We define

$$\delta_x^{(\alpha)} U_m = -\frac{1}{h^\alpha}\sum_{k=0}^{M} g_{m-k}^{(\alpha)} U_k.$$

Lemma (Macías-Díaz)

If $\alpha \in (1,2]$ then $\langle -\delta_x^{(\alpha)} U, V\rangle = \langle \delta_x^{(\alpha/2)} U, \delta_x^{(\alpha/2)} V\rangle$, for any $U, V \in \mathcal{V}_h$. Moreover, if $\alpha \in (1,2]$ and $g_h^{(\alpha)} = 2g_0^{(\alpha)} h^{1-\alpha}$, then

1. $\|\delta_x^{(\alpha/2)} V\|_2^2 \leq g_h^{(\alpha)} \|V\|_2^2$, *for each $V \in \mathcal{V}_h$, and*
2. $\|\delta_x^{(\alpha)} V\|_2^2 \leq (g_h^{(\alpha)} \|V\|_2)^2$, *for each $V \in \mathcal{V}_h$.*

Definition

Let $(U^n)_{n\in\bar{I}_N} \subseteq \mathcal{V}_h$ and $G : \mathbb{R}^{p+2} \to \mathbb{R}$. We define

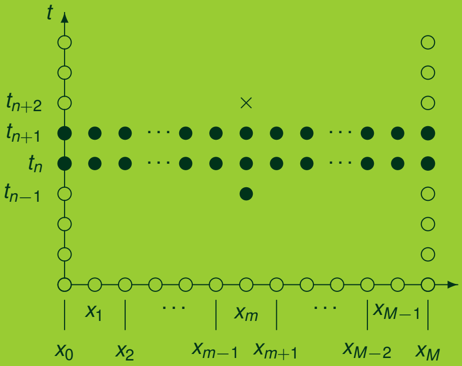
$$\delta_U G(U_m^{n+\frac{1}{2}}) = \begin{cases} \dfrac{G(U_m^{n+1}) - G(U_m^n)}{U_m^{n+1} - U_m^n}, & \text{if } U_m^{n+1} \neq U_m^n, \\ G'(U_m^{n+\frac{1}{2}}), & \text{if } U_m^{n+1} = U_m^n, \end{cases}$$

and let $\delta_{\delta_x^{(\alpha_j/2)}U} G_j(U_m^{n+\frac{1}{2}}) = \delta_{\delta_x^{(\alpha_j/2)}U} G_j(\delta_x^{(\alpha_j/2)} U_m^{n+\frac{1}{2}})$.

The following is the **FDS** to solve the **IBVP**:

$$\boxed{\begin{aligned} &\hat{\delta}_t^{(2)} U_m^{n+\frac{1}{2}} + \gamma \delta_t U_m^{n+\frac{1}{2}} + \delta_U G(U_m^{n+\frac{1}{2}}) = \\ &\qquad = \sum_{j=1}^{p} \delta_x^{(\alpha_j/2)} \left[\delta_{\delta_x^{(\alpha_j/2)}U} G_j(U_m^{n+\frac{1}{2}}) \right], \\ &\text{such that} \begin{cases} U_m^0 = \phi_U(x_m), & \forall m \in I_{M-1}, \\ \delta_t \mu_t U_m^0 = \psi_U(x_m), & \forall m \in I_{M-1}, \\ U_0^n = U_M^n = 0, & \forall n \in \bar{I}_N. \end{cases} \end{aligned}}$$

Stencil



Theorem (Existence and uniqueness)

If $G, G_1, G_2, \ldots, G_p \in \mathcal{C}^1(\mathbb{R})$, then the FDS has a unique solution. $\square$

We define the **discrete energy density** as

$$\boxed{H_m^n = \frac{1}{2}(\delta_t U_m^{n+\frac{1}{2}})(\delta_t U_m^{n-\frac{1}{2}}) + G(U_m^n) + \sum_{j=1}^{p} G_j(\delta_x^{(\alpha_j/2)} U_m^n).}$$

The **discrete total energy** at the time t_n is given by $\boxed{E^n = h \sum_{m=0}^{M} H_m^n.}$

Lemma

If $(U^n) \subseteq \mathcal{V}_h$, then

1. $\langle \hat{\delta}_t^{(2)} U^{n+\frac{1}{2}}, \delta_t U^{n+\frac{1}{2}} \rangle = \frac{1}{2} \delta_t \langle \delta_t U^{n+\frac{1}{2}}, \delta_t U^{n-\frac{1}{2}} \rangle$;
2. $\langle \delta_t U^{n+\frac{1}{2}}, \delta_t U^{n-\frac{1}{2}} \rangle = \mu_t \|\delta_t U^{n-\frac{1}{2}}\|_2^2 - \frac{1}{2}\tau^2 \|\delta_t^{(2)} U^n\|_2^2$;
3. $\langle \delta_U G(U^{n+\frac{1}{2}}), \delta_t U^{n+\frac{1}{2}} \rangle = \delta_t \langle G(U^{n+\frac{1}{2}}), 1 \rangle$; *and*
4. $\langle -\delta_x^{(\alpha_j/2)} [\delta_{\delta_x^{(\alpha_j/2)} U} G_j(U_m^{n+\frac{1}{2}})], \delta_t U^{n+\frac{1}{2}} \rangle = \delta_t \langle G_j(\delta_x^{(\alpha_j/2)} U^{n+\frac{1}{2}}), 1 \rangle$.

Theorem (Dissipation of energy)

Let $(U^n) \subseteq \mathcal{V}_h$ be solution of the FDS. Then $\delta_t E^{n+\frac{1}{2}} = -\gamma \|\delta_t U^{n+\frac{1}{2}}\|_2^2$.

Proof.

Multiply both sides of the FDS by $\delta_t U^{n+\frac{1}{2}}$, use the identities in the previous lemma and collect terms to check that

$$\begin{aligned} 0 &= \frac{1}{2}\delta_t \langle \delta_t U^{n+\frac{1}{2}}, \delta_t U^{n-\frac{1}{2}} \rangle + \gamma \|\delta_t U^{n+\frac{1}{2}}\|_2^2 + \delta_t \langle G(U^{n+\frac{1}{2}}), 1 \rangle \\ &\quad + \sum_{j=1}^{p} \delta_t \langle G_j(\delta_x^{(\alpha_j/2)} U^{n+\frac{1}{2}}), 1 \rangle \\ &= \delta_t E^{n+\frac{1}{2}} + \gamma \|\delta_t U^{n+\frac{1}{2}}\|_2^2, \end{aligned}$$

whence the result readily follows. □

Contents

Let $u_m^n = u(x_m, t_n)$, and define the operators $\mathcal{D}$ and D, by

$$\boxed{\begin{aligned}\mathcal{D}u(x,t) = \frac{\partial^2 u(x,t)}{\partial t^2} + \gamma \frac{\partial u(x,t)}{\partial t} + \frac{dG(u(x,t))}{dv} \\ - \sum_{j=1}^{p} \frac{\partial^{\alpha_j/2}}{\partial |x|^{\alpha_j/2}} \frac{dG_j}{dv} \left(\frac{\partial^{\alpha_j/2} u(x,t)}{\partial |x|^{\alpha_j/2}} \right),\end{aligned}}$$

and

$$\boxed{\begin{aligned}Du_m^n = \hat{\delta}_t^{(2)} u_m^{n+\frac{1}{2}} + \gamma \delta_t u_m^{n+\frac{1}{2}} + \delta_u G(u_m^{n+\frac{1}{2}}) \\ - \sum_{j=1}^{p} \delta_x^{(\alpha_j/2)} \left[\delta_{\delta_x^{(\alpha_j/2)} u} G_j(u_m^{n+\frac{1}{2}}) \right],\end{aligned}}$$

Definition

If $U = (U^n) \subseteq \mathcal{V}_h$, then we define $|||U|||_\infty = \max\{\|U^n\|_\infty : n \in \bar{I}_M\}$.

Theorem (Consistency)

Let $u \in \mathcal{C}^{5,4}_{x,t}(\overline{\Omega})$, $G \in \mathcal{C}^2(\mathbb{R})$ and $G_1, G_2, \dots, G_p \in \mathcal{C}^5(\mathbb{R})$. There exist constants $C, C' \in \mathbb{R}^+$, which are independent of h and τ, such that

$$|||\mathcal{D}u - Du|||_\infty \leq C(\tau^2 + h^2),$$

$$|||\mathcal{H}u - Hu|||_\infty \leq C'(\tau + h^2).$$

We will require the following results in the sequel.

1. If $U, V \in \mathcal{V}_h$, then $2|\langle U, V\rangle| \leq \|U\|_2^2 + \|V\|_2^2$.
2. If $n \in \mathbb{N}$ and $U^1, U^2, \dots, U^n \in \mathcal{V}_h$, then
$$\|U^1 + U^2 + \ldots + U^n\|_2^2 \leq n(\|U^1\|_2^2 + \|U^2\|_2^2 + \ldots + \|U^n\|_2^2).$$
3. If $(U^n)_{n=0}^N \subseteq \mathcal{V}_h$ and $n \in \{1, \dots, N\}$, then
$$\|U^n\|_2^2 \leq 2\|U^0\|_2^2 + 2T\tau \sum_{k=0}^{n-1} \|\delta_t U^k\|_2^2.$$

Let $\Phi_U = (\phi_U, \psi_U)$, where $\phi_U, \psi_U : B \to \mathbb{R}$ and $U = V, W$.

Lemma (Macías-Díaz)

Let $G \in \mathcal{C}^2(\mathbb{R})$ and $(R^{n+\frac{1}{2}})_{n=0}^{N-1} \subseteq \mathcal{V}_h$. Let $(V^n)_{n=0}^N$ and $(W^n)_{n=0}^N$ be solutions of the FDS for Φ_V and Φ_W, respectively. Let $\varepsilon^n = V^n - W^n$,

$$\Lambda_{V,W} G^{n+\frac{1}{2}} = \delta_V G(V^{n+\frac{1}{2}}) - \delta_W G(W^{n+\frac{1}{2}}).$$

Then there exists $C' \in \mathbb{R}^+ \cup \{0\}$ which depends only on G, such that

$$\begin{aligned} 4\tau \sum_{n=1}^{k} \left| \langle R^{n+\frac{1}{2}} - \Lambda_{V,W} G^{n+\frac{1}{2}}, \delta_t \varepsilon^{n+\frac{1}{2}} \rangle \right| \leq{}& 4\tau \sum_{n=0}^{k} \| R^{n+\frac{1}{2}} \|_2^2 + C' \|\varepsilon^0\|_2^2 \\ &+ C'\tau \sum_{n=0}^{k} \|\delta_t \varepsilon^{n+\frac{1}{2}}\|_2^2, \end{aligned}$$

$$4k\tau^2 \sum_{n=1}^{k} \|\Lambda_{V,W} G^{n+\frac{1}{2}}\|_2^2 \leq C' T^2 \|\varepsilon^0\|_2^2 + C' T^3 \tau \sum_{n=0}^{k} \|\delta_t \varepsilon^{n+\frac{1}{2}}\|_2^2.$$

Lemma

Let $G_j \in \mathcal{C}^2(\mathbb{R})$ for each j, and let $(V^n)_{n=0}^N$ and $(W^n)_{n=0}^N$ be solutions of the FDS for Φ_V and Φ_W, respectively. Let $\varepsilon^n = V^n - W^n$, and define

$$\Theta_{V,W} G_j^{n+\frac{1}{2}} = \delta_x^{(\alpha_j/2)} \left[\delta_{\delta_x^{(\alpha_j/2)} V} G_j(V^{n+\frac{1}{2}}) \right] - \delta_x^{(\alpha_j/2)} \left[\delta_{\delta_x^{(\alpha_j/2)} W} G_j(W^{n+\frac{1}{2}}) \right].$$

There is $C'' \in \mathbb{R}^+ \cup \{0\}$ depending only on G, such that for each k,

$$4\tau \sum_{n=1}^{k} \sum_{j=1}^{p} \left| \langle \Theta_{V,W} G_j^{n+\frac{1}{2}}, \delta_t \varepsilon^{n+\frac{1}{2}} \rangle \right| \leq C'' \|\varepsilon^0\|_2^2 + C'' \tau \sum_{n=0}^{k} \|\delta_t \varepsilon^{n+\frac{1}{2}}\|_2^2,$$

$$4pk\tau^2 \sum_{n=1}^{k} \sum_{j=1}^{p} \|\Theta_{V,W} G_j^{n+\frac{1}{2}}\|_2^2 \leq C'' T^2 \|\varepsilon^0\|_2^2 + C'' T^3 \tau \sum_{n=0}^{k} \|\delta_t \varepsilon^{n+\frac{1}{2}}\|_2^2.$$

Proof.

By the lemma, there is $C_j' \in \mathbb{R}^+ \cup \{0\}$ depending on G_j, such that

$$4\tau \sum_{n=1}^{k} \left| \langle \Theta_{V,W} G_j^{n+\frac{1}{2}}, \delta_t \varepsilon^{n+\frac{1}{2}} \rangle \right| =$$

$$= 4\tau \sum_{n=1}^{k} \left| \left\langle \Lambda_{\delta_x^{(\alpha_j/2)} V, \delta_x^{(\alpha_j/2)} W} G_j^{n+\frac{1}{2}}, \delta_t \delta_x^{(\alpha_j/2)} \varepsilon^{n+\frac{1}{2}} \right\rangle \right|$$

$$\leq C_j' \| \delta_x^{(\alpha_j/2)} \varepsilon^0 \|_2^2 + C_j' \tau \sum_{n=0}^{k} \| \delta_t \delta_x^{(\alpha_j/2)} \varepsilon^{n+\frac{1}{2}} \|_2^2$$

$$\leq C_j'' \| \varepsilon^0 \|_2^2 + C_j'' \tau \sum_{n=0}^{k} \| \delta_t \varepsilon^{n+\frac{1}{2}} \|_2^2,$$

whence the conclusion readily follows. □

Lemma (Pen-Yu)

Let $(\omega^n)_{n=0}^N, (\rho^n)_{n=0}^N \subseteq \mathbb{R}^+ \cup \{0\}$, and let $C \geq 0$ satisfy

$$\omega^k \leq \rho^k + C\tau \sum_{n=0}^{k-1} \omega^k, \quad \forall k \in I_{N-1}.$$

Then $\omega^n \leq \rho^n e^{Cn\tau}$ for each $n \in \bar{I}_N$.

Theorem (Stability)

Let $G, G_1, G_2, \ldots, G_p \in \mathcal{C}^2(\mathbb{R})$. Let Φ_V and Φ_W be two sets of initial conditions for the **FDS**, *and let $(V^n)_{n\in\bar{I}_N}$ and $(W^n)_{n\in\bar{I}_N}$ be the respective solutions. Let $\varepsilon^n = V^n - W^n$, and define*

$$\omega^k = \|\delta_t \varepsilon^{k+\frac{1}{2}}\|_2^2,$$

$$\rho = C_1(1 + T^2)\|\varepsilon^0\|_2^2 + 2\mu_t\|\delta_t\varepsilon^{\frac{1}{2}}\|_2^2 + 4T^2\|\delta_t^{(2)}\varepsilon^1\|_2^2.$$

There exists $C \in \mathbb{R}^+ \cup \{0\}$ independent of τ, such that $\omega^k \leq \rho e^{CT}$.

Theorem (Convergence)

Let $u \in \mathcal{C}^{5,4}_{x,t}(\overline{\Omega})$ be the solution of **IBVP**, *and suppose that $G \in \mathcal{C}(\mathbb{R})$ and $G_1, G_2, \ldots, G_p \in \mathcal{C}^5(\mathbb{R})$. Then the numerical solution converges to the exact one with $\mathcal{O}(\tau^2 + h^2)$.*

Proof.

Let $\epsilon^n = u^n - U^n$, and notice that

$$\hat{\delta}_t^{(2)} \epsilon^{n+\frac{1}{2}} + \gamma \delta_t \epsilon^{n+\frac{1}{2}} + \Lambda_{u,U} G^{n+\frac{1}{2}} - \sum_{j=1}^{p} \Theta_{u,U} G_j^{n+\frac{1}{2}} = R_m^{n+\frac{1}{2}},$$

$$\text{such that } \begin{cases} \epsilon_m^0 = \delta_t \mu_t \epsilon_m^0 = 0, & \forall m, \\ \epsilon_0^n = \epsilon_M^n = 0, & \forall n, \end{cases}$$

where $R_m^{n+\frac{1}{2}}$ is the local truncation error, and $|||R|||_\infty \leq C_0(\tau^2 + h^2)$, for some $C_0 \geq 0$. Now, take inner product with $\delta_t \epsilon^{n+\frac{1}{2}}$ **(to be continued).**

Proof (continued).

We obtain

$$\begin{aligned}
\|\delta_t\epsilon^{k+\frac{3}{2}}\|_2^2 \leq 2\mu_t\|\delta_t\epsilon^{\frac{1}{2}}\|_2^2 + \tau^2\|\delta_t^{(2)}\epsilon^{k+1}\|_2^2 \\
&+ 4\tau\sum_{n=1}^{k}\left|\langle R^{n+\frac{1}{2}} - \Lambda_{V,W}G^{n+\frac{1}{2}}, \delta_t\epsilon^{n+\frac{1}{2}}\rangle\right| \\
&+ 4\tau\sum_{n=1}^{k}\sum_{j=1}^{p}\left|\langle \Theta_{V,W}G_j^{n+\frac{1}{2}}, \delta_t\epsilon^{n+\frac{1}{2}}\rangle\right| \\
&\leq 2\mu_t\|\delta_t\epsilon^{\frac{1}{2}}\|_2^2 + \tau^2\|\delta_t^{(2)}\epsilon^{k+1}\|_2^2 + 4\tau\sum_{n=0}^{k}\|R^{n+\frac{1}{2}}\|_2^2 \\
&+ C_1\|\epsilon^0\|_2^2 + C_1\tau\sum_{n=0}^{k}\|\delta_t\epsilon^{n+\frac{1}{2}}\|_2^2.
\end{aligned}$$

We estimate now the term $\tau^2\|\delta_t^{(2)}\epsilon^{k+1}\|_2^2$ **(to be continued).**

Proof (continued).

Using induction, we prove that

$$\delta_t^{(2)}\epsilon^{k+1} = (-1)^k\delta_t^{(2)}\epsilon^1 - \sum_{n=1}^{k}(-1)^{n+k}\left[\gamma\delta_t\epsilon^{n+\frac{1}{2}} + \Lambda_{V,W}G^{n+\frac{1}{2}} - \sum_{j=1}^{p}\Theta_{V,W}G_j^{n+\frac{1}{2}} + R^{n+\frac{1}{2}}\right].$$

Using now Young's inequality

$$\tau^2\|\delta_t^{(2)}\epsilon^{k+1}\|_2^2 \le 5T^2\|\delta_t^{(2)}\epsilon^1\|_2^2 + 5\gamma^2 T\tau\sum_{n=1}^{k}\|\delta_t\epsilon^{n+\frac{1}{2}}\|_2^2 + C_1' T^2\|\epsilon^0\|_2^2 + C_1' T^3\tau\sum_{n=0}^{k}\|\delta_t\epsilon^{n+\frac{1}{2}}\|_2^2 + 5T\tau\sum_{n=1}^{k}\|R^{n+\frac{1}{2}}\|_2^2$$

(to be continued).

Proof (continued).

Substituting and regrouping, we reach

$$\omega^{k+1} \leq \rho^{k+1} + C_2 \tau \sum_{n=0}^{k} \omega^k, \qquad \forall k \in I_{N-2},$$

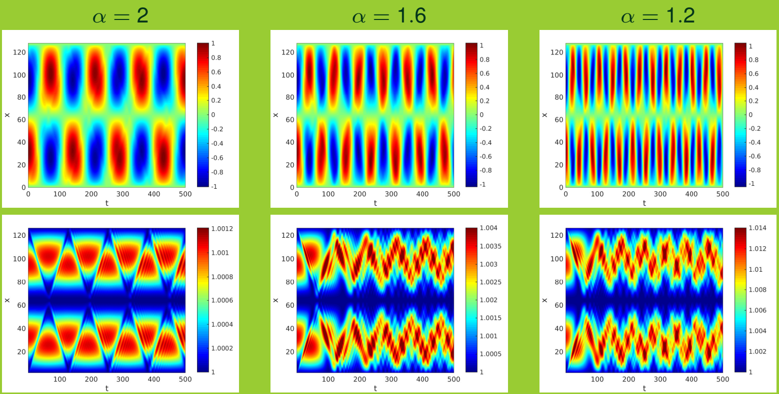
where $C_2 = C_1 + C_1' T^3 + 5\gamma^2 T$, and

$$\begin{aligned} \omega^k &= \|\delta_t \epsilon^{k+\frac{1}{2}}\|_2^2, \\ \rho^k &= (C_1 + C_1' T^2)\|\epsilon^0\|_2^2 + 2\mu_t \|\delta_t \epsilon^{\frac{1}{2}}\|_2^2 + 5T^2 \|\delta_t^{(2)} \epsilon^1\|_2^2 \\ &\quad + (4 + 5T)\tau \sum_{n=0}^{k} \|R^{n+\frac{1}{2}}\|_2^2. \end{aligned}$$

Pen-You's lemma yields $\omega^k \leq C_3 \rho^k$. Then $\|\delta_t \epsilon^{k+\frac{1}{2}}\|_2^2 \leq C_4^2(\tau^2 + h^2)^2$, which yields $\|\epsilon^n\|_2 \leq C(\tau^2 + h^2)$. $\square$

Contents

Undamped elastic string



Elastic string

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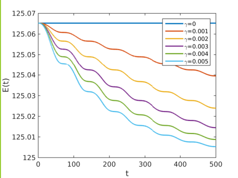

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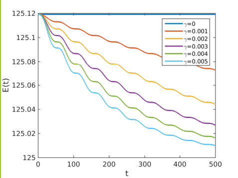

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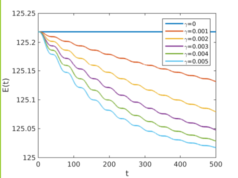

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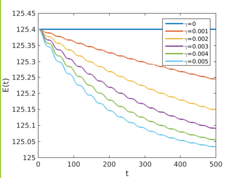

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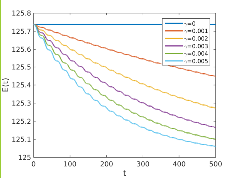

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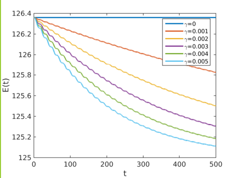
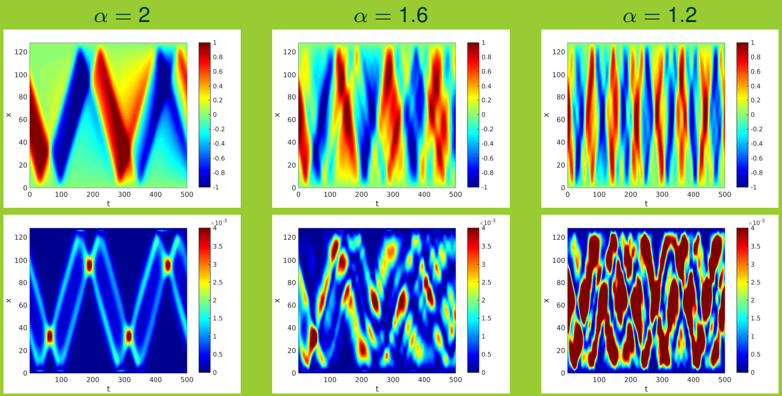


Undamped FPUT



FPUT

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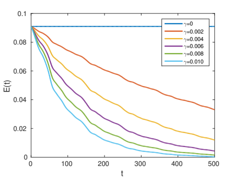

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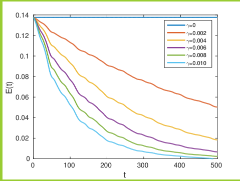

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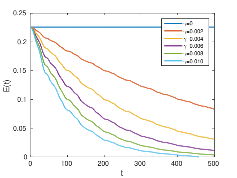

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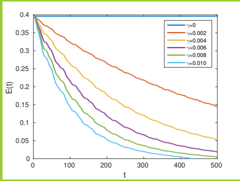


Temporal convergence analysis						
	$h = 1$		$h = 0.5$		$h = 0.25$	
τ	$\epsilon_{t,h}$	ρ_τ	$\epsilon_{t,h}$	ρ_τ	$\epsilon_{t,h}$	ρ_τ
$0.1/2^0$	2.6805×10^{-2}	–	6.7596×10^{-3}	–	1.6854×10^{-3}	–
$0.1/2^1$	6.4470×10^{-3}	2.0558	1.6881×10^{-3}	2.0015	4.2704×10^{-4}	1.9807
$0.1/2^2$	1.5595×10^{-3}	2.0475	4.1406×10^{-4}	2.0275	1.0711×10^{-4}	1.9952
$0.1/2^3$	3.8014×10^{-4}	2.0365	9.8730×10^{-5}	2.0683	2.6528×10^{-5}	2.0136
$0.1/2^4$	9.1697×10^{-5}	2.0516	2.3789×10^{-5}	2.0532	6.4825×10^{-6}	2.0329

Spatial convergence analysis						
	$\tau = 5 \times 10^{-5}$		$\tau = 2.5 \times 10^{-5}$		$\tau = 1.25 \times 10^{-5}$	
h	$\epsilon_{t,h}$	ρ_h	$\epsilon_{t,h}$	ρ_h	$\epsilon_{t,h}$	ρ_h
$4/2^0$	7.1887×10^{-3}	–	1.8178×10^{-3}	–	4.5327×10^{-4}	–
$4/2^1$	1.8250×10^{-3}	1.9778	5.0045×10^{-4}	1.8609	1.2482×10^{-4}	1.8605
$4/2^2$	4.5941×10^{-4}	1.9900	1.2724×10^{-4}	1.9756	3.1762×10^{-5}	1.9745
$4/2^3$	1.0807×10^{-4}	2.0877	3.2302×10^{-5}	1.9779	8.0961×10^{-6}	1.9720
$4/2^4$	2.0730×10^{-5}	2.3822	7.8347×10^{-6}	2.0437	1.9029×10^{-6}	2.0890

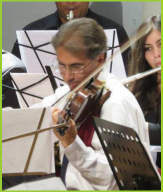
Conclusions

- A fractional extension of the FPUT model was proposed.
- A conserved energy functional was derived.
- A conservative scheme was proposed and fully analyzed.
- This is the first work in which these tasks have been achieved.
- It is easy to extend the study to multiple dimensions.

Thanks to . . .

- Organization of this event.
- National Council of Science and Techology (CONACYT).

Thank y'all!



Prof. J. E. Macías-Díaz, PhD
Editor, *Intern. J. Comput. Math.* (Taylor & Francis)
Adv. Math. Phys. (Hindawi)
Comput. Math. Meth. (Wiley)
Appl. Numer. Math. (Elsevier)
Open Physics (De Gruyter)
Email: `jemaciasdiaz@hotmail.com`
ResearchGate: J_Macias-Diaz
Publons: 1329933
Orcid: 0000-0002-7580-7533,
ResearcherID: H-8635-2018
Scholar: J. E. Macías-Díaz