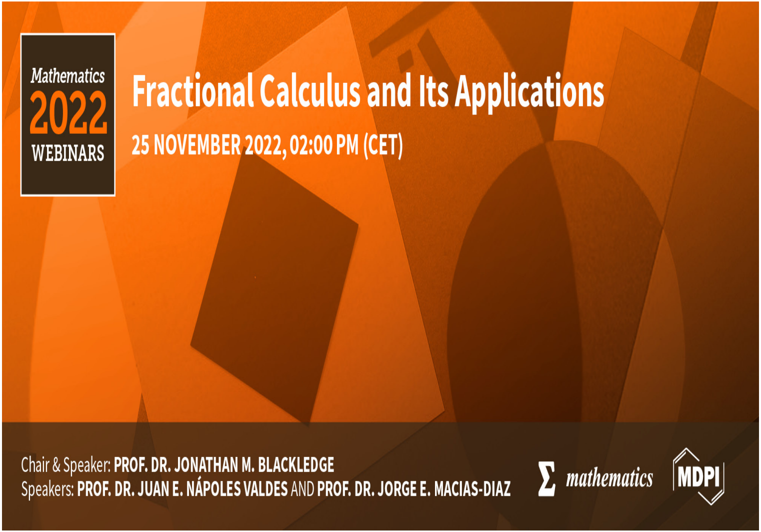
Acknowledgment



From L'Hopital to Atangana: Some Applications of Fractional Calculus

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Fractional Calculus and Its Applications
25 November 2022

Resumen

In this presentation, we will touch on some historical aspects of Fractional Calculus, its main exponents, and some of the most recent applications, mainly linked to Integral Inequalities.

Introduction

September 30, 1695, is a wonderful date for Mathematical Sciences!

To encourage comprehension of the subject, we present some fractional integral operators (with $0 \leq a < t < b \leq \infty$). The first is the classic Riemann-Liouville fractional integrals ([49]).

Definition

Let $\psi \in L_1[a,b]$. Then the Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ are defined by (right and left respectively):

$$I^{\alpha}_{\nu_1+}\psi(x) = \frac{1}{\Gamma(\alpha)} \int_{\nu_1}^{x} (x-t)^{\alpha-1}\psi(t)\,dt, \quad x > \nu_1 \quad (0.1)$$

$$I^{\alpha}_{\nu_2-}\psi(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\nu_2} (t-x)^{\alpha-1}\psi(t)\,dt, \quad x < \nu_2. \quad (0.2)$$

The left-sided and right-sided Riemann-Liouville k-fractional integrals are given in [50].

Definition

Let $f \in L_1[a,b]$. Then the Riemann-Liouville k-fractional integrals of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ and $k > 0$ are given by the expressions:

$$ {}^{\alpha}I^{k}_{a^+} f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_a^u (u-\tau)^{\frac{\alpha}{k}-1} f(\tau)\, d\tau, \quad u > a, \tag{0.3} $$

$$ {}^{\alpha}I^{k}_{b^-} f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_u^b (\tau-u)^{\frac{\alpha}{k}-1} f(\tau)\, d\tau, \quad u < b. \tag{0.4} $$

Another known fractional integral is as follows (see [34] and [41]).

Definition

Let $f \in L^1[a,b];\mathbb{R}), (a,b) \in \mathbb{R}^2, a < b$. The right and life side Hadamard fractional integrals of order α with Re$(\alpha) > 0$ are defined by

$$H^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (log\frac{t}{s})^{\alpha-1} \frac{f(s)}{s} ds, \quad a < t < b, \tag{0.5}$$

and

$$H^{\alpha}_{b^-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (log\frac{s}{t})^{\alpha-1} \frac{f(s)}{s} ds, \quad a < t < b. \tag{0.6}$$

In [38], the author introduced new fractional integral operators, called the Katugampola fractional integrals, in the following way (also see [18]):

Definition

Let $0 < a < b < +\infty$, $f : [a, b] \to \mathbb{R}$ is an integrable function, and $\alpha \in (0, 1)$ and $\rho > 0$ two fixed real numbers. The right and life side Katugampola fractional integrals of order α are defined by

$$K_{a^+}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} f(s)ds, \quad a < t \tag{0.7}$$

and

$$K_{b^-}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{t^{\rho-1}}{(s^\rho - t^\rho)^{1-\alpha}} f(s)ds, \quad t < b. \tag{0.8}$$

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A more general definition of the Riemann-Liouville fractional integrals is given in [41].

Definition

Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a,b]$ with a continuous derivative g' on (a,b). The left and right sided fractional integrals of a function f with respect to another function g on $[a,b]$ of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ are expressed by:

$$ {}_{g}^{\alpha}I_{a^+} f(u) = \frac{1}{\Gamma(\alpha)} \int_a^u \big(g(u) - g(\tau)\big)^{\alpha-1} g'(\tau) f(\tau)\, d\tau, \quad u > a, \tag{0.9} $$

$$ {}_{g}^{\alpha}I_{b^-} f(u) = \frac{1}{\Gamma(\alpha)} \int_u^b \big(g(\tau) - g(u)\big)^{\alpha-1} g'(\tau) f(\tau)\, d\tau, \quad u < b. \tag{0.10} $$

A k-fractional analogue of the Definition 5 is given in the following (see [2], [45] and [61]):

Definition

Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a, b]$ with a continuous derivative g' on (a, b). The left and right sided k-fractional integrals of a function f with respect to another function g on $[a, b]$ of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ and $k > 0$ are expressed by:

$$ {}^{\alpha}I^{k}_{a^+} f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_a^u \big(g(u) - g(\tau)\big)^{\frac{\alpha}{k}-1} g'(\tau) f(\tau)\, d\tau, \quad u > a, \tag{0.11} $$

$$ {}^{\alpha}I^{k}_{b^-} f(u) = \frac{1}{k\Gamma_k(\alpha)} \int_u^b \big(g(\tau) - g(u)\big)^{\frac{\alpha}{k}-1} g'(\tau) f(\tau)\, d\tau, \quad u < b, \tag{0.12} $$

First generalization:

Definition

Let $0 < a < b < +\infty$, $\phi : [a,b] \to \mathbb{R}$ is an integrable function, and $\alpha \in (0,1)$. Let $F(t)$ be an integrable function on $[a,b]$. The right and life side weighted fractional integrals of order α are defined by

$$J^F_{a+}\phi(t) = \int_a^t \frac{\phi(s)}{(\mathbb{F}(t-s))^{1-\alpha}} \frac{ds}{F(s)}, \quad a < t \tag{0.13}$$

and

$$J^F_{b-}\phi(t) = \int_t^b \frac{\phi(s)}{(\mathbb{F}(s-t))^{1-\alpha}} \frac{ds}{F(s)}, \quad t < b, \tag{0.14}$$

with $\mathbb{F}(u-v) = \int_v^u F(z)dz$.

Definition

The Caputo derivative of fractional order α of function $f(t)$ (right and left) are defined as

$$ {}^{C}\mathrm{D}^{\alpha}_{a+}(f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(t-s)^{\alpha}} ds, \quad t < b, \tag{0.15} $$

$$ {}^{C}\mathrm{D}^{\alpha}_{b-}(f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_t^b \frac{f'(s)}{(s-t)^{\alpha}} ds, \quad a < t, \tag{0.16} $$

with $0 < \alpha \leq 1$.

Remark

There is not much difficulty in defining the higher orders.

An extension of ${}^{C}D^{\alpha}$ is the so-called Caputo-Fabricio derivative (see [7, 16]), given by:

$$ {}^{CF}D_a^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(s) e^{-\frac{\alpha(t-s)}{1-\alpha}}\, ds, \qquad (0.17)$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$.

A more recent extension is the *Atangana-Baleanu derivative*, defined in [4] by

$$ {}^{ABC}D_a^\alpha f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(s) E_\alpha\Big(- \frac{\alpha (t-s)^\alpha}{1-\alpha} \Big) ds, \tag{0.18} $$

where

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} $$

is the *Mittag-Leffler function*.

Second generalization ([53]):

Definition

The right and left sided wieghted Caputo derivative of fractional order α of function $f(t)$ is defined as

$$ {}^{C}\mathrm{D}^{\alpha}_{a+}(f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t f'(s)K(t,s)ds, \quad t < b \tag{0.19} $$

$$ {}^{C}\mathrm{D}^{\alpha}_{b-}(f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_t^b f'(s)K(s,t)ds, \quad a < t \tag{0.20} $$

with $0 < \alpha \leq 1$.

Remark

Let us see the following particular cases.
Put $K(t,s) = \frac{1}{\Gamma(1-\alpha)(t-s)^{\alpha}}$ *we obtain the right Caputo derivative (similarly to the left sided).*
With $K(t,s) = \frac{M(\alpha)}{(1-\alpha)} E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right]$ *we obtain the right Atangana-Baleanu-Caputo derivative (the obtaining of the left sided is leaved to the readers).*
Considering $K(t,s) = \frac{w(t)^{-1}}{\Gamma(1-\alpha)}(w(s)f(s))'\,(z(t)-z(s))^{-\alpha}$ *we obtain the left/forward weighted/scaled derivative of order* α *of a function* $f(t)$ *with respect to another function* $z(t)$ *and weight* $w(t)$ *([1]) obviously in a similar way we obtain the other side derivative.*

What advantages does the above general definition offer? First, a compact and simple way of working and, second, it allows you to choose the kernel depending on the problem to be solved. It is clear that taking into account the previous integral operators, we can define new Caputo-type differential operators, with an adequate choice of kernel.

It is clear that the formulations corresponding to the derivatives of the Riemann-Liouville type, is an unsolved problem.

Some applications to Integral Inequalities

In what follows, I is a real, closed and bounded interval. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on the interval I, if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{0.21}$$

holds for all $x, y \in I$ and $t \in [0,1]$. We say that f is concave if $-f$ is convex.

Hermite-Hadamard Inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{0.22}$$

is true for any convex function f en $[a, b]$.
This inequality was published by Hermite in 1883 ([36]) and, independently, by Hadamard in 1893 ([34]). It gives an estimate of the mean value of a convex function and observes that it also provides an analysis of the inequality of Jensen.

1) Using different notions of convexity.
2) Refinement of the mesh used.
3) Improvement of the estimates of the left and right members of (0.22).
4) Using new generalized and fractional integral operators.

In [6] we presented the following definitions.

Definition

Let $h : [0,1] \to \mathbb{R}$ be a nonnegative function, $h \neq 0$. The nonnegative function $\psi : [0,+\infty) \to [0,+\infty)$ is said to be (h,m)-convex modified of first type on $[0,+\infty)$ if inequality

$$\psi\left(tx + m(1-t)y\right) \leq h^s(t)\psi(x) + m(1-h^s(t))\psi(y) \qquad (0.23)$$

is fulfilled for $m \in [0,1]$, $s \in [-1,1]$, for all $x, y \in I$ and $t \in [0,1]$.

Definition

Let $h : [0, 1] \to \mathbb{R}$ nonnegative functions, $h \neq 0$. The nonnegative function $\psi : [0, +\infty) \to [0, +\infty)$ is said to be (h, m)-convex modified of second type on $[0, +\infty)$ if inequality

$$\psi\left(tx + m(1 - t)y\right) \leq h^s(t)\psi(x) + m(1 - h(t))^s\psi(y) \qquad (0.24)$$

is fulfilled for $m \in [0, 1]$, $s \in [-1, 1]$, for all $x, y \in I$ and $t \in [0, 1]$.

From Definitions 10 and 11 we can define $N^s_{h,m}[a,b]$ as the set of functions (h,m)-convex modified, for which $f(a) \geq 0$, characterized by the triple $(h(t), m, s)$. Note that if:
$(h(t), 0, 0)$ we have the increasing functions ([9]).
$(t, 0, s)$ we have the s-starshaped functions ([9]).
$(t, 0, 1)$ we have the starshaped functions ([9]).
$(t, 1, 1)$ then ψ is a convex function on $[0, +\infty)$ ([9]).
$(t, m, 1)$ then ψ is a m-convex function on $[0, +\infty)$ ([63]).
$(t, 1, s)$ $s \in (0, 1]$ then ψ is a s-convex function on $[0, +\infty)$.
$(t, 1, s)$ $s \in [-1, 1]$ then ψ is a s-convex function on $[0, +\infty)$.
$(t^a, 1, s)$ with $a \in (0, 1]$, then ψ is a $s-(a, m)$-convex function on $[0, +\infty)$.
$(h(t), m, 1)$ then ψ is an (h, m)-convex function on $[0, +\infty)$.

Final Remarks

Usually convex functions are presented with this argument

$$f\left(ta+(1-t)b\right)$$

Let's think in a more general way, let's put it like this

$$f\left(\frac{t}{n+1}a+(\frac{n+1-t}{n+1})b\right)$$

if $n = 0$, it is obvious that the first is obtained, but by taking different values of n, we can generate different points in the interval $[a, b]$, which affects the generality of the results obtained.

Next we present the weighted integral operators, other possible line work.

Definition

Let $\phi \in L\left([a,b]\right)$ and let w be a continuous and positive function, $w : [0,1] \to [0,+\infty)$, with first order derivatives integrable on I. Then the weighted fractional integrals are defined by (right and left respectively):

$$J^{w}_{a^+}\phi(r) = \int_a^r w\left(\frac{\sigma - a}{b-a}\right)\phi(\sigma)d\sigma$$

and

$$J^{w}_{b^-}\phi(r) = \int_r^b w\left(\frac{b-\sigma}{b-a}\right)\phi(\sigma)d\sigma,$$

with $a < r \le b$ and k some real number.

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