Mathematics 2022 WEBINARS

Fractional Calculus and Its Applications

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Fractional Calculus and Fractal Geometry:
A Unification using Einstein's Evolution Equation

MDPI

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Σ mathematics

Einstein's Evolution Equation:
What is it and where does it come from?

1905: Einstein publishes a paper which is the first of its type to model quantitatively, the nature of Brownian motion due to the random movement of atoms (before atomic theory was fully established)

INVESTIGATIONS ON THE THEORY OF ,THE BROWNIAN MOVEMENT

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Putting for the number of particles per unit volume $\nu = f(x, t)$, we will calculate the distribution of the particles at a time $t + \tau$ from the distribution at the time t. From the definition of the function $+(\Delta)$, there is easily obtained the number of the particles which are located at the. time $t + \tau$ between two planes perpendicular to the x-axis, with abscissæ x and $x + dx$. We get

$$\boxed{f(x, t+\tau)dx = dx. \int_{\Delta=-\infty}^{\Delta=+\infty} f(x+\Delta)\phi(\Delta)d\Delta.}$$

Primary aim of the presentation in a nutshell:

To show how this equation can be used to derive fractional order field equations whose solutions exhibit self-affine (fractal) characteristics

Contents

Einstein's Evolution Equation in 3D

$$u(\mathbf{r}, t+\tau) = u(\mathbf{r}, t) \otimes p(\mathbf{r}) + s(\mathbf{r}, t), \quad \mathbf{r} \in \mathbb{R}^3$$

$u(\mathbf{r}, t)$ - **Density Function**
The number of particles undergoing a random walk per unit volume.

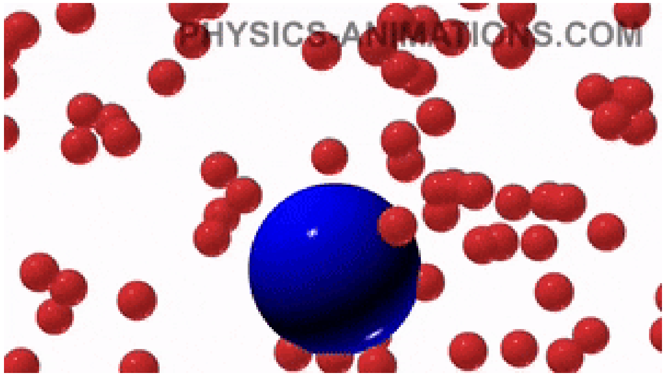
$s(\mathbf{r}, t)$ - **Source Function**
Stochastic function that 'drives' the random motion of particles.

$p(\mathbf{r})$ - **Probability Density Funtion**
Governs the distribution to which any and all particles conform.

$\otimes$ - **Convolution Integral Operator**

Using the **Convolution Theorem** we can write this equation in **Fourier space** as

$$U(\mathbf{k}, t+\tau) = U(\mathbf{k}, t)P(\mathbf{k}) + S(\mathbf{k}, t)$$



The Kolmogorov-Feller Equation

Based on applying a Taylor series in time to the evolution equation, and applying a trick!

$$u(\mathbf{r}, t+\tau) = p(\mathbf{r}) \otimes u(\mathbf{r}, t) + s(\mathbf{r}, t)$$

$$u(\mathbf{r}, t+\tau) = u(\mathbf{r}, t) + \tau \frac{\partial}{\partial t} u(\mathbf{r}, t) + \frac{\tau^2}{2!} \frac{\partial^2}{\partial t^2} u(\mathbf{r}, t) + \ldots$$

Memory Function

$$= u(\mathbf{r}, t) + \tau m(t) \otimes \frac{\partial}{\partial t} u(\mathbf{r}, t)$$

$$\boxed{\tau m(t) \otimes \frac{\partial}{\partial t} u(\mathbf{r}, t) = -u(\mathbf{r}, t) + p(\mathbf{r}) \otimes u(\mathbf{r}, t) + s(\mathbf{r}, t)}$$

Evolution Equation for a Memory Function

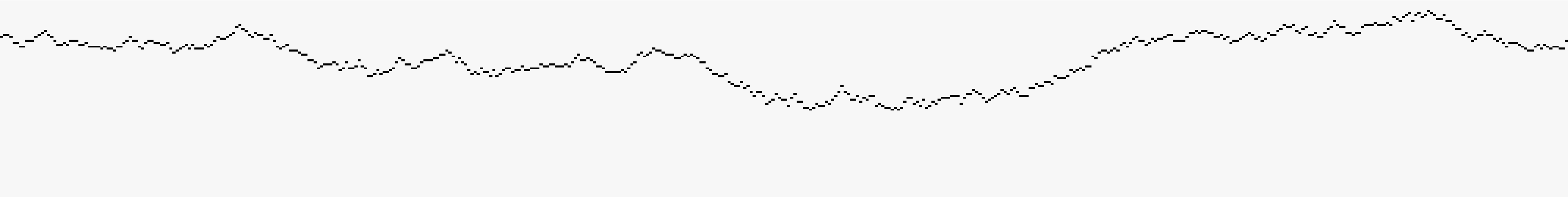
Time Series Analysis

$$\boxed{\tau m(t) \otimes \frac{\partial}{\partial t} u(\mathbf{r}, t) = -u(\mathbf{r}, t) + p(\mathbf{r}) \otimes u(\mathbf{r}, t) + s(\mathbf{r}, t)}$$

- Let $p(\mathbf{r}) = \delta^3(\mathbf{r})$ so that $\tau m(t) \otimes \dfrac{\partial}{\partial t} u(t) = s(t)$

- **Memory Function** that generates a **self-affine** time series is

$$m(t) = \frac{1}{\Gamma(1-\alpha)\, t^{\alpha}},\ t > 0,\ \alpha \in [0,1] \ \Rightarrow \boxed{\lambda^{\alpha} \Pr[u(t)] = \Pr[u(\lambda t)]}$$



Model for the **Fractal Market Hypothesis**, e.g. www.mdpi.com/2227-7390/7/11/1057

On the Laplacian operator

- Consider the Gaussian Characteristic Function

$$P(\mathbf{k}) = \exp(-ck^2) \sim 1 - ck^2, \quad c << 1$$

- Then, since

$$p(\mathbf{r}) \otimes u(\mathbf{r}, t) \;\leftrightarrow\; P(\mathbf{k})U(\mathbf{k}, t)$$

Real Space **Fourier Space**

Pierre-Simon Laplace
(1749-1827)

we have

Laplacian

$$U(\mathbf{k}, t)P(\mathbf{k}) =$$

$$U(\mathbf{k}, t) - ck^2 U(\mathbf{k}, t) \leftrightarrow u(\mathbf{r}, t) + c\nabla^2 u(\mathbf{r}, t)$$

Derivation of the (classical) wave equation

$$\boxed{\tau m(t) \otimes \frac{\partial}{\partial t} u(\mathbf{r}, t) = -u(\mathbf{r}, t) + p(\mathbf{r}) \otimes u(\mathbf{r}, t) + s(\mathbf{r}, t)}$$

- Let $m(t) = \frac{d}{dt}\delta(t)$ and $P(\mathbf{k}) = \exp(-ck^2),\ c << 1$

- Yields the (linear) **Wave Equation** (with source function $s = 0$)

$$\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} u(\mathbf{r}, t) = \nabla^2 u(\mathbf{r}, t), \quad c_0 = \sqrt{\frac{c}{\tau}}$$

Laplacian ⟷ **Evolution Equation** ⟷ **Wave Mechanics**

Derivation of the Schrodinger equation

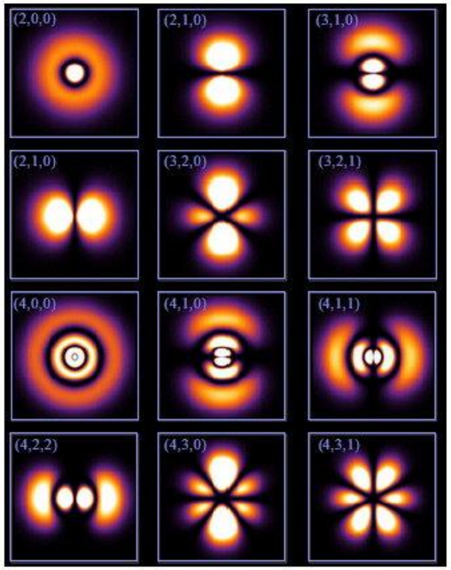
$$\boxed{\tau m(t) \otimes \frac{\partial}{\partial t} u(\mathbf{r}, t) = -u(\mathbf{r}, t) + p(\mathbf{r}) \otimes u(\mathbf{r}, t) + s(\mathbf{r}, t)}$$

- Let $P(\mathbf{k}) = \exp(-ck^2),\ c << 1$ (with source function $s = 0$)

$$\tau m(t) = -i\hbar\delta(t),\ c = \frac{\hbar^2}{2m} \quad \Rightarrow \quad \boxed{i\hbar\frac{\partial}{\partial t} u(\mathbf{r}, t) = -\frac{\hbar^2}{2m}\nabla^2 u(\mathbf{r}, t)}$$

- Note that for the rest mass of an electron

$$c \sim 5 \times 10^{-37} << 1$$



Field Equations for a Levy Distribution

- What happens if we consider the generalised Characteristic Function (for Levy Index γ):

$$P(\mathbf{k}) = \exp(-c \mid k \mid^{\gamma}); \;\; \gamma \in [0,2], \;\; c << 1$$

- In this case, using the **Riesz definition** for a fractional Laplacian

Fractional Laplacian

$$U(\mathbf{k},t)P(\mathbf{k}) =$$

$$U(\mathbf{k},t) - c \mid k \mid^{\gamma} U(\mathbf{k},t) \leftrightarrow u(\mathbf{r},t) + c\nabla^{\gamma} u(\mathbf{r},t)$$

Fractional Laplacian and Wave Mechanics: Mandelbrot Surfaces

- For a Levy distributed process, the wave equation becomes

$$\left(\nabla^{\gamma} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right) u(\mathbf{r}, t) = -s(\mathbf{r}, t), \; c_0 = \sqrt{\frac{c}{\tau}}$$

- For wave mechanics taking place over very large time scales

$$\boxed{\nabla^{\gamma} u(\mathbf{r}) = -s(\mathbf{r})}$$

Fractional Poisson Equation

- For $\mathbf{r} \in \mathbb{R}^2$ and a **white noise** source function, the solution to this equation is a ***Mandelbrot surface***, for a **Fractal Dimension** D where

$$\gamma = 4 - D, \; D \in [2, 3]$$

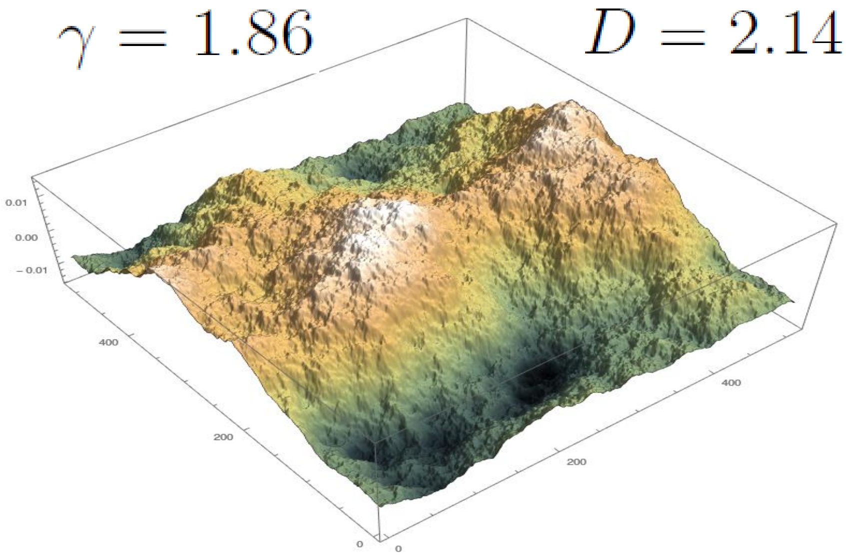
Example: Modelling the Himalayas

- The Himalayas are the result of **Plate Tectonics**: the movement of plates across the earths surface due to internal heat plumes.

- Deformation of rock is taken to be due to the wave mechanics of very viscous fluids.

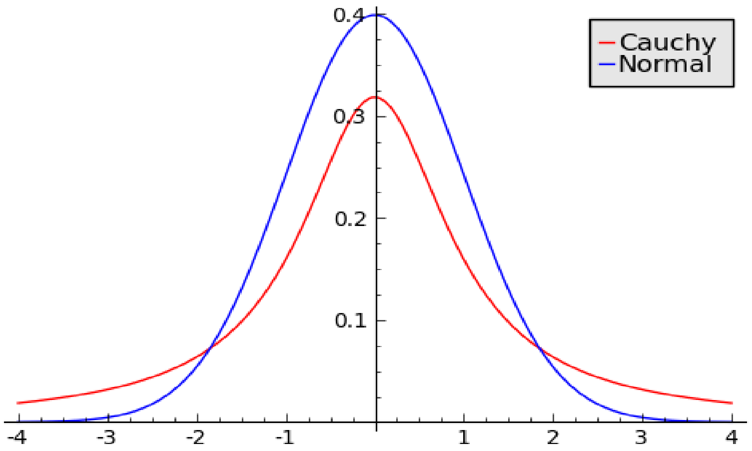
- If the evolution is taken to be a very slow Levy distributed process, then, at a given point in time, we can model a local area of the Himalayas in terms of a Mandelbrot surface (for a fixed Fractal Dimension) conforming to the fractional Poisson equation

$$\nabla^{\gamma} u(\mathbf{r}) = -s(\mathbf{r}) \;\Rightarrow \mid \mathbf{k} \mid^{\gamma} U(\mathbf{k}) = S(\mathbf{k})$$



Fractional Diffusion and the Continuity Equations

For $m(t) = \delta(t)$



$$\frac{\partial}{\partial t} u(\mathbf{r}, t) = D \nabla^{\gamma} u(\mathbf{r}, t)$$

Gaussian PDF $\quad \gamma = 2 \swarrow \qquad \searrow \gamma = 1 \quad$ Cauchy PDF

$$\boxed{\frac{\partial}{\partial t} u(\mathbf{r}, t) = D \nabla^2 u(\mathbf{r}, t)} \qquad \boxed{\frac{\partial}{\partial t} u(\mathbf{r}, t) = D \hat{\mathbf{n}} \cdot \nabla u(\mathbf{r}, t)}$$

$\Downarrow$ **Diffusion Equation** $\qquad$ $\Downarrow$ **Continuity Equation**

Fractional Diffusion and the ***Biodynamics Hypothesis***

- There is no such thing as ‘perfect diffusion’ or ‘perfect propagation’, and the equations thereof, that provide ‘perfect models’ for these phenomena.

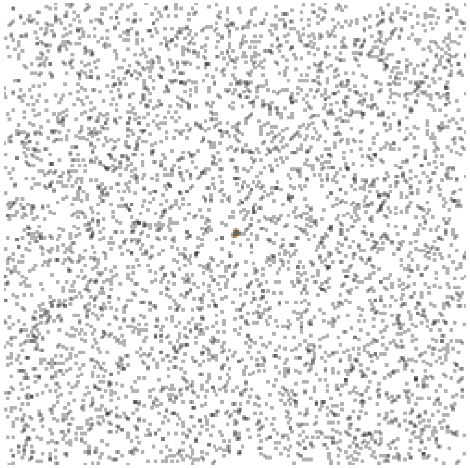
- In reality there is always a mixture of both, i.e.

Fractional Diffusion/Propagation

⬍

Diffusion Mechanics coupled with **Wave Mechanics**

- Basis for the **Biodynamic Hypothesis:**



DNA replication (directional) + Brownian motion
(Long chain molecule + Water)

= Fractional Diffusion ➡ **Fractal Geometry**

Fractional Quantum Mechanics

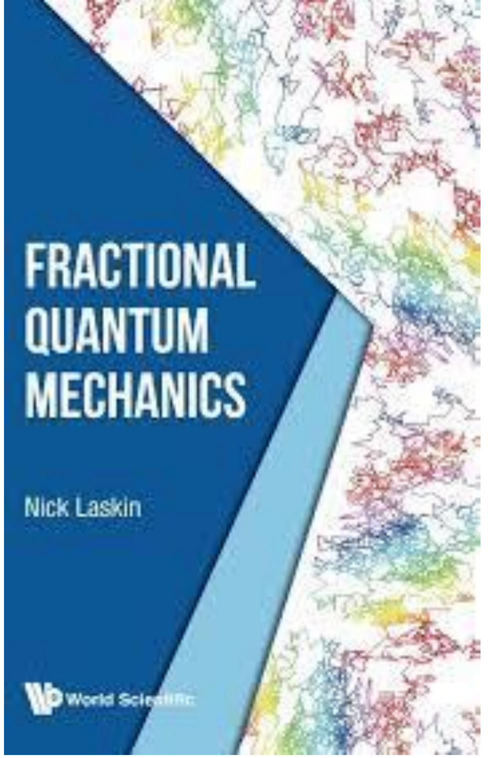
- Concerned with the equations of quantum mechanics involving **fractional derivatives**

- Examples (for Planck Units and mass *m*):

  - **Fractional Schrodinger Equation**

$$i\frac{\partial}{\partial t}u(\mathbf{r},t) = -\frac{1}{2m}\nabla^{\gamma}u(\mathbf{r},t)$$ (for a delta memory function)

  - **Fractional Schrodinger-Klein-Gordon Equation**



$$\left[\left(\frac{1}{2m}\right)^{1-\alpha}\nabla^2 + \frac{\partial^{1+\alpha}}{\partial(-it)^{1+\alpha}} - \alpha m^2\right]u(\mathbf{r},t) = 0;\ \alpha\in[0,1]$$

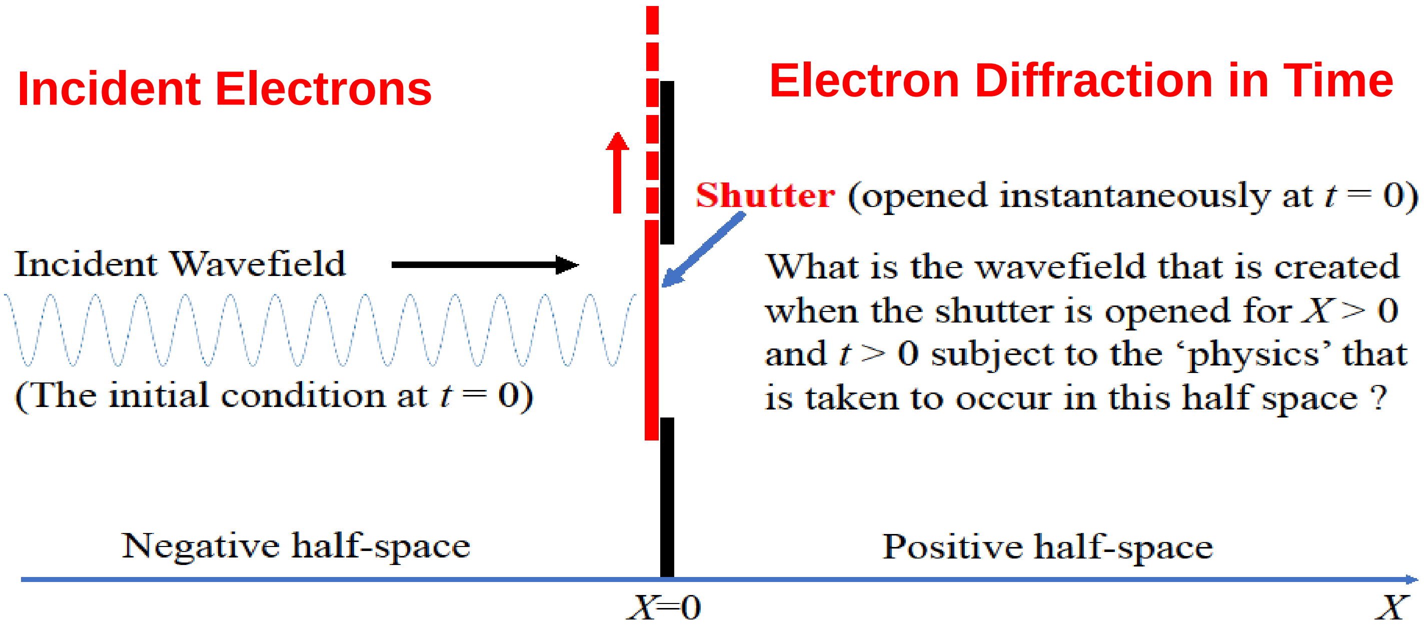
$$\Rightarrow \begin{cases} \left(\frac{1}{2m}\nabla^2 + i\frac{\partial}{\partial t}\right)u(\mathbf{r},t) = 0, & \alpha = 0; \\ \left(\nabla^2 - \frac{\partial^2}{\partial t^2} - m^2\right)u(\mathbf{r},t) = 0, & \alpha = 1. \end{cases}$$

Schrodinger Equation ($\alpha = 0$)

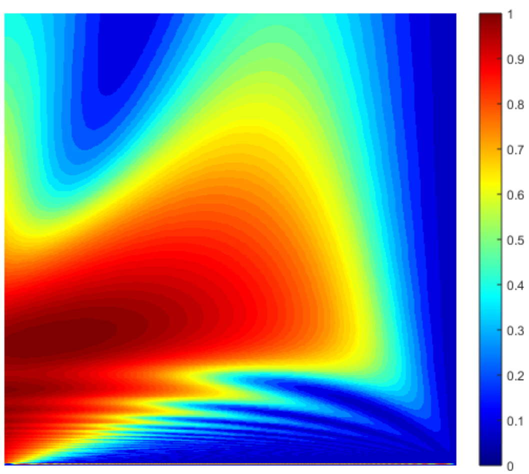
Klein-Gordon Equation ($\alpha = 1$)

Example publication: https://arrow.tudublin.ie/engscheleart2/303/

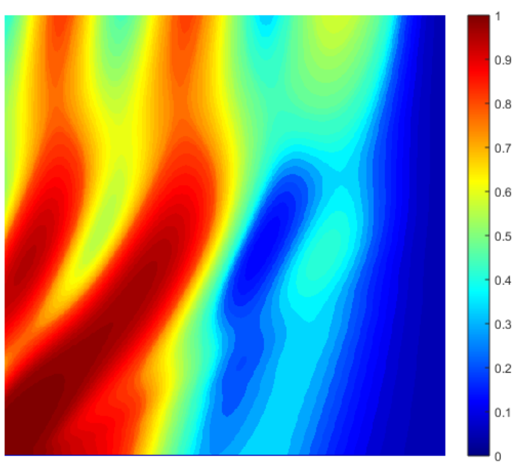
The Fractional Quantum Shutter Problem



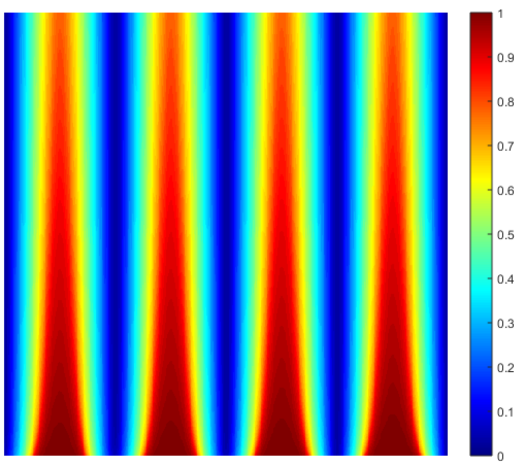
Example space (horizontal) - time (vertical) maps



Non-relativistic



Semi-relativistic



Fully-relativistic

Summary

$$\boxed{\tau m(t) \otimes \frac{\partial}{\partial t} u(\mathbf{r}, t) = -u(\mathbf{r}, t) + p(\mathbf{r}) \otimes u(\mathbf{r}, t) + s(\mathbf{r}, t)}$$

- In the context of the evolution equation, the Laplacian is a consequence of considering an evolutionary process to be governed by a **normal (Gaussian) distribution**.

- The inclusion of fields that are taken to evolve through **non-Gaussian** (Levy) processes, and **non-delta** type memory functions, can yield models for the **Fractal Geometry of Nature** based on the solutions of fractional differential equations.

- The evolution equation provides a basis for constructing field equations that are intrinsic to stochastic processes – **the random motion of atoms.**

A Final Thought:

How many definitions of a fractional differo-integral are there?

- **Valerio et al.,** ***How Many Fractional Derivatives Are There?*** **MDPI Mathematics, 2022** [Online] Available at: https://www.mdpi.com/2227-7390/10/5/737/htm

- **Blackledge**, 2020 [Online] Available at: https://arrow.tudublin.ie/engscheleart2/245/ develops another **generalised definition** based on a convolution integral of the form

$$f^{(\gamma)}(x) = \frac{1}{2} f(x) \otimes \mathrm{sgn}^{(1+\gamma)}(x), \ \gamma \in (-\infty, \infty)$$

- This can be used to generate various approximations for a fractional differential using a range of sigmoid-type functions to approximate the sign function.

- It leads to the following **conjecture**: **There is no upper bound to the number of self-consistent definitions that can be composed, to specify a fractional differential.**

- The conjecture is based on an assumed unlimited number of linear operators whose eigenfunctions are complete, when, for any such function $\phi(x)$ and with $0 < \alpha < 1$,

$$f^{(\alpha)}(x) = f(x) \otimes \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \sum_{n=0}^{\infty} \phi_n(x) \int_a^x \frac{\phi_n^*(y)}{(x-y)^\alpha} dy$$

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