Tomonaga-Luttinger liquid universality

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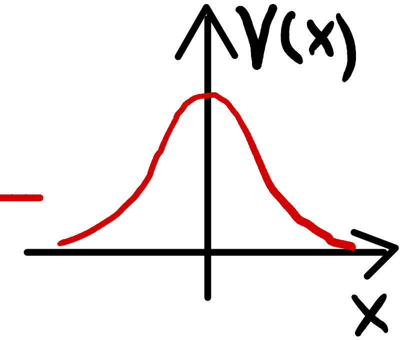
The Tomonaga-Luttinger model is the low-energy fixed-point model under renormalization group flow of a large class of gapless (metallic) one-dimensional models of correlated fermions (and bosons)

$\longrightarrow$ Tomonaga-Luttinger liquid universality

(Haldane `81)

Typical microscopic models



- electron gas with screened Coulomb interaction ($\hbar = 1$):

$$\hat{H} = \sum_{k,\sigma} \frac{k^2}{2m} \hat{c}^\dagger_{k,\sigma} \hat{c}_{k,\sigma} + \frac{1}{2} \int dx \int dx' \hat{\rho}(x) V(x - x') \hat{\rho}(x')$$

- Hubbard model:

$$\hat{H} = -t \sum_{j,\sigma} \left(\hat{c}^\dagger_{j,\sigma} \hat{c}_{j+1,\sigma} + \text{H.c.} \right) + U \sum_j \hat{n}_{j,\uparrow} \hat{n}_{j,\downarrow}$$

- spinless fermions with nearest neighbor interaction:

$$\hat{H} = -t \sum_{j} \left(\hat{c}^\dagger_{j} \hat{c}_{j+1} + \text{H.c.} \right) + U \sum_j \hat{n}_{j} \hat{n}_{j+1}$$

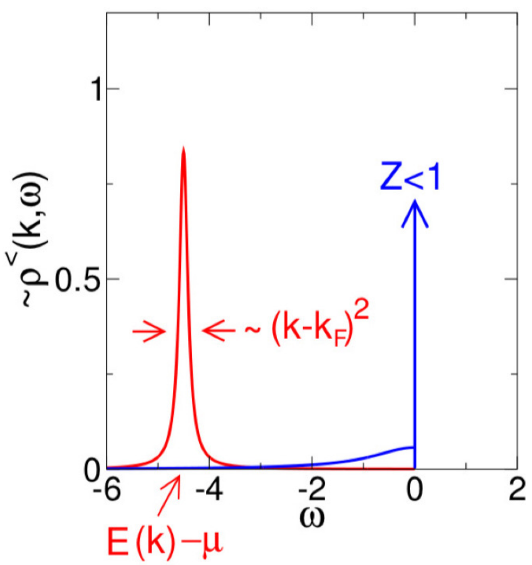
$\longrightarrow$ the two lattice models are **integrable**

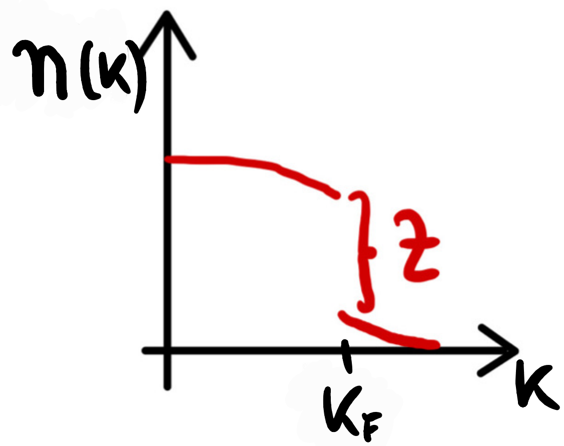
Interacting metals in 3d—Fermi liquids

- N-particle ground state is very complicated
- fundamental insight: (Landau '56)

  1-to-1-correspondence of the low-lying excited states of the interacting and the non-interacting system - quasi-particles

- e.g. low-temperature specific heat: $C_V = \frac{m^*}{m}\gamma_0 T$
- photoemission spectra (at $T = 0$):

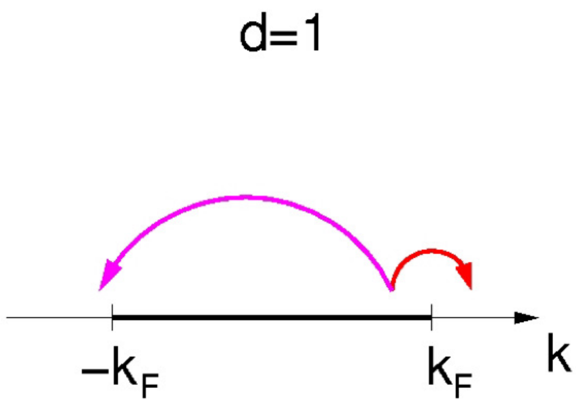


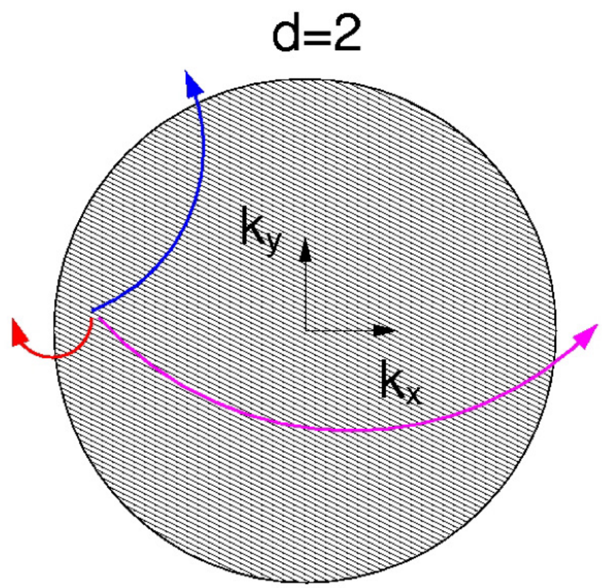


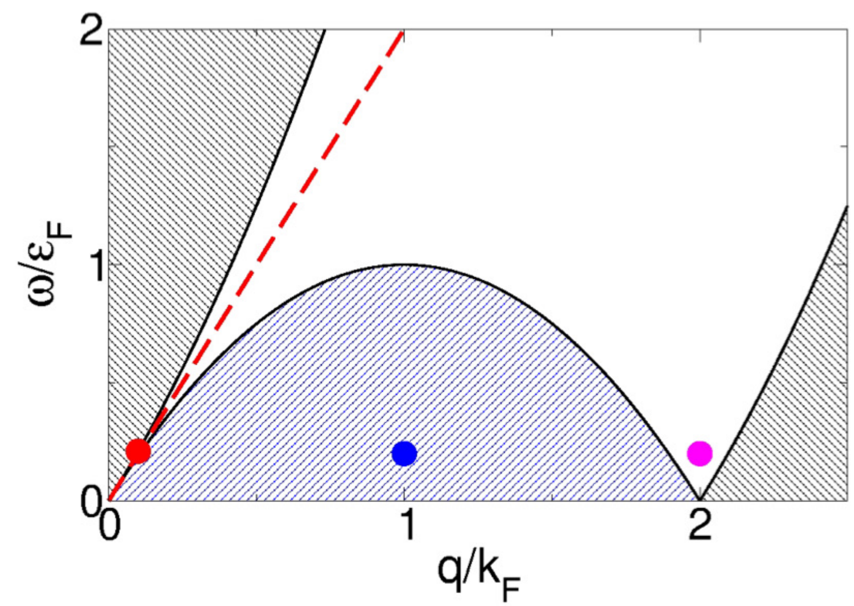
- quasi-particles: renormalized mass m^*, weight $Z < 1$
- jump height Z in momentum distribution n(k) at k_F

$\longrightarrow$ the free electron gas is the low-energy fixed point model

Excitations in 1d — non-interacting electrons



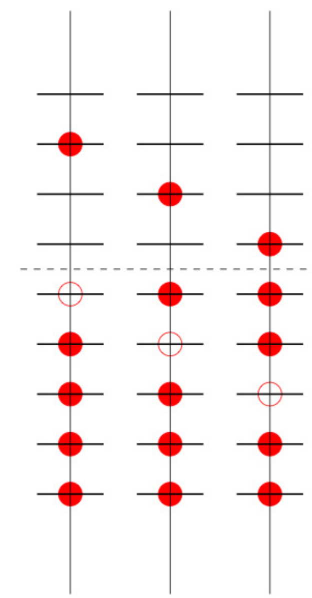




- small ω and q: unique relation $\omega(q) = v_F q$

- linear combination of many degenerate particle-hole excitations?

- collective excitations of bosonic nature?



Add interaction and look closer: Tomonaga-Luttinger model

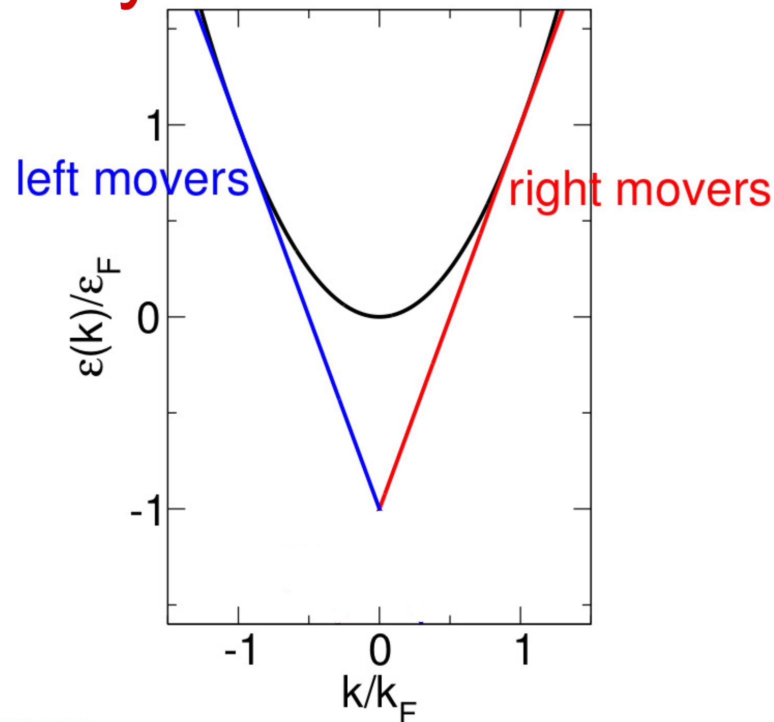
- spinless interacting electron gas: ground state has admixture of p-h excitations
- consider the high density limit ($\tilde{v}(q) \approx 0$ for $q > q_c$): $k_F \gg q_c$

decays

- interaction sufficiently weak

$\Rightarrow$ ground state and low-energy excited states: no holes deep in Fermi sea and particles with $|k| - k_F \gg q_c$

$\Rightarrow$ linearize dispersion around $\pm k_F$



- kinetic energy:

$$\hat{T} = \sum_{n \geq 0} v_F (k_n - k_F) \, \hat{c}_n^\dagger \hat{c}_n + \sum_{n<0} (-v_F)(k_n + k_F) \, \hat{c}_n^\dagger \hat{c}_n$$

- screened Coulomb interaction in momentum space:

$$\hat{V} = \frac{1}{2L} \sum_{n \neq 0} \tilde{v}(q_n) \hat{\rho}_n^\dagger \hat{\rho}_n + \frac{1}{2L} \hat{N}^2 \tilde{v}(0)$$

is finite

- density:

$$\hat{\rho}_n = \sum_{n' \geq 0} \hat{c}_{n'}^\dagger \hat{c}_{n'+n} + \sum_{n'<0} \hat{c}_{n'}^\dagger \hat{c}_{n'+n} = \hat{\rho}_{n,+} + \hat{\rho}_{n,-} \, , \quad \hat{\rho}_n^\dagger = \hat{\rho}_{-n}$$

(Tomonaga ´50)

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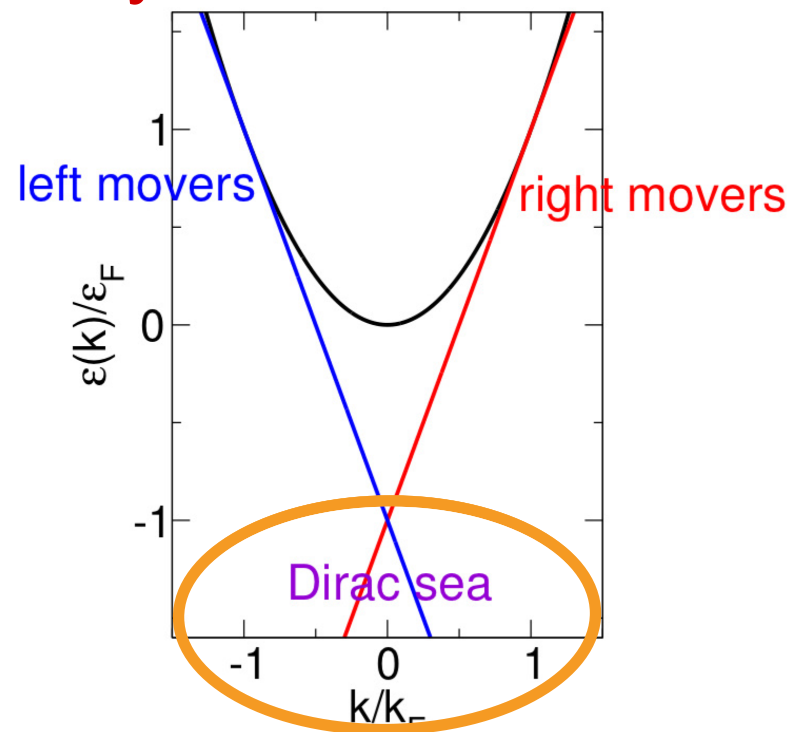


- kinetic energy:

$$\hat{T} = \sum_{\cancel{n>0}} v_F(k_n - k_F)\,\hat{c}_n^\dagger \hat{c}_n + \sum_{\cancel{n<0}} (-v_F)(k_n + k_F)\,\hat{c}_n^\dagger \hat{c}_n$$

added

- screened Coulomb interaction in momentum space:

$$\hat{V} = \frac{1}{2L}\sum_{n\neq 0} \tilde{v}(q_n)\hat{\rho}_n^\dagger \hat{\rho}_n + \frac{1}{2L}\hat{N}^2\tilde{v}(0)$$

ultra-violet?

?

is finite

- density:

$$\hat{\rho}_n = \sum_{\cancel{n'>0}} \hat{c}_{n'}^\dagger \hat{c}_{n'+n} + \sum_{\cancel{n'<0}} \hat{c}_{n'}^\dagger \hat{c}_{n'+n} = \hat{\rho}_{n,+} + \hat{\rho}_{n,-}\,, \quad \hat{\rho}_n^\dagger = \hat{\rho}_{-n}$$

(Luttinger `63)

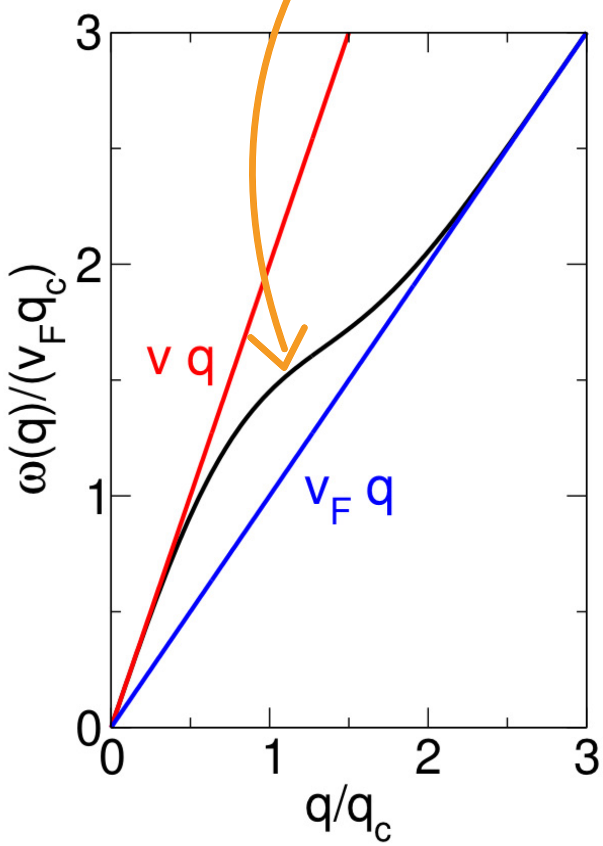
Tomonaga-Luttinger model — the bosonized Hamiltonian

- Hamiltonian in the bosons:

$$\hat{H} = \sum_{n>0} q_n \left\{ \left(v_F + \frac{\tilde{v}(q_n)}{2\pi} \right) \left(\hat{b}_n^\dagger \hat{b}_n + \hat{b}_{-n}^\dagger \hat{b}_{-n} \right) + \frac{\tilde{v}(q_n)}{2\pi} \left(\hat{b}_n^\dagger \hat{b}_{-n}^\dagger + \hat{b}_{-n} \hat{b}_n \right) \right\}$$
$$+ \frac{\pi}{2L} \left(v_N \hat{\mathcal{N}}^2 + v_J \hat{\mathcal{J}}^2 \right) = \hat{H}_b + \hat{H}_{\mathcal{N},\mathcal{J}}$$

- $\left(\hat{\mathcal{N}}_+ + \hat{\mathcal{N}}_-\right)^2 = \hat{\mathcal{N}}^2 \qquad \left(\hat{\mathcal{N}}_+ - \hat{\mathcal{N}}_-\right)^2 = \hat{\mathcal{J}}^2$

non-linear!



- two velocities: $v_N = v_F + \tilde{v}(0)/\pi$, $v_J = v_F$
- diagonalize: $\hat{\alpha}_n^\dagger = c_n \hat{b}_n^\dagger + s_n \hat{b}_{-n}$

$$\hat{H}_b = \sum_{n \neq 0} \omega(q_n) \hat{\alpha}_n^\dagger \hat{\alpha}_n + \text{const.}$$

- dispersion of eigenmodes:

$$\omega(q_n) = v_F |q_n| \sqrt{1 + \tilde{v}(q_n)/(\pi v_F)} \overset{q \to 0}{\approx} v|q_n|$$

$$v = \sqrt{v_N \, v_J}$$

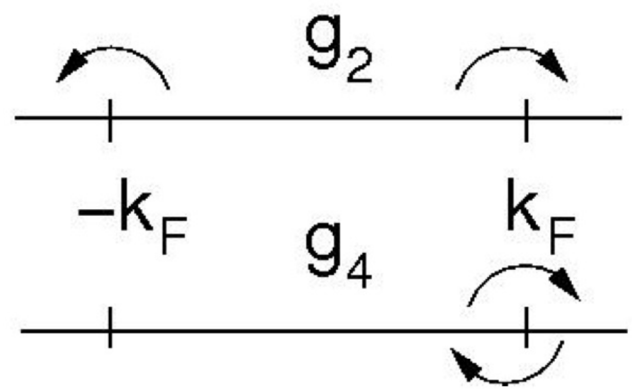
Towards the Tomonaga-Luttinger liquid— g-ology

class of models

- generalized Hamiltonian:

$$\hat{H}_{\mathrm{TL}} = \sum_{n>0} q_n \left\{ \left(v_F + \frac{g_4(q_n)}{2\pi} \right) \left(\hat{b}_n^\dagger \hat{b}_n + \hat{b}_{-n}^\dagger \hat{b}_{-n} \right) + \frac{g_2(q_n)}{2\pi} \left(\hat{b}_n^\dagger \hat{b}_{-n}^\dagger + \hat{b}_{-n} \hat{b}_n \right) \right\}$$

$$+ \frac{\pi}{2L} \left(v_N \hat{\mathcal{N}}^2 + v_J \hat{\mathcal{J}}^2 \right)$$



- forward scattering: inter-branch g_2

  intra-branch g_4

- two velocities: $v_N = v_F + [g_4(0) + g_2(0)]/(2\pi), \quad v_J = v_F + [g_4(0) - g_2(0)]/(2\pi)$

- diagonalization $\hat{\alpha}_n^\dagger = c_n \hat{b}_n^\dagger + s_n \hat{b}_{-n}$: $\hat{H}_b = \sum_{n\neq 0} \omega(q_n) \hat{\alpha}_n^\dagger \hat{\alpha}_n$ + const.

$$v_{N/J}(q_n) = v_F + [g_4(q_n) \pm g_2(q_n)]/(2\pi) \approx v_{N/J}$$

$$\omega(q_n) = |q_n| \sqrt{v_J(q_n)\, v_N(q_n)} = |q_n| v(q_n) \approx v|q_n|$$

$$K_n = \sqrt{v_J(q_n)/v_N(q_n)} \approx K \quad \Rightarrow K_n = 1 \text{ for vanishing interaction}$$

$$c_n = \frac{1}{2}\left(\sqrt{K_n} + \frac{1}{\sqrt{K_n}} \right) \approx c\,, \; s_n = \frac{1}{2}\left(\sqrt{K_n} - \frac{1}{\sqrt{K_n}} \right) \approx s$$

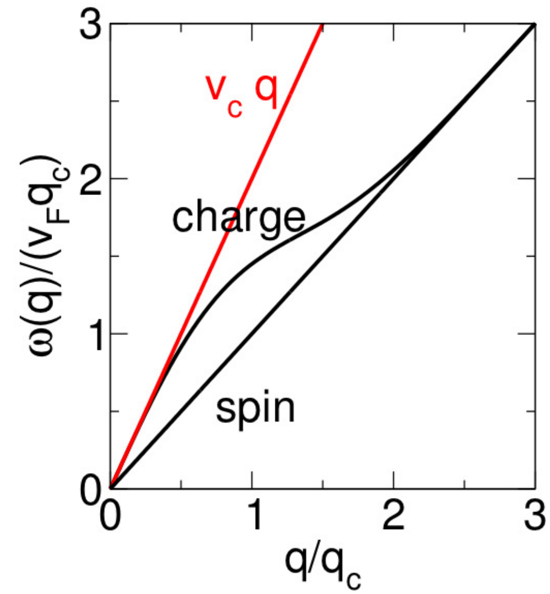
Finally: include the spin (which is straight forward)

- $\hat{\rho}_{n,\pm} \longrightarrow \hat{\rho}_{n,\pm,\sigma} \quad \Rightarrow \quad \hat{b}_{n,\sigma}$
- $g_{2/4}(q) \longrightarrow g_{2/4}^{\sigma,\sigma'}(q) = \delta_{\sigma,\sigma'}\, g_{2/4,\parallel}(q) + \delta_{\sigma,-\sigma'}\, g_{2/4,\perp}(q)$
- $\hat{b}_{n,c/s} = \frac{1}{\sqrt{2}}\left(\hat{b}_{n,\uparrow} \pm \hat{b}_{n,\downarrow}\right)$
- $g_{\nu,c/s}(q) = g_{\nu,\parallel}(q) \pm g_{\nu,\perp}(q)$ with $\nu = 2, 4$
- $\hat{\mathcal{N}}_{\alpha,c/s} = \frac{1}{\sqrt{2}}\left(\hat{\mathcal{N}}_{\alpha,\uparrow} \pm \hat{\mathcal{N}}_{\alpha,\downarrow}\right)$ with $\alpha = +, -$

$$\Rightarrow \hat{H}_{\mathrm{TL}}^{(1/2)} = \hat{H}_{\mathrm{TL},c} + \hat{H}_{\mathrm{TL},s} \text{ with } \left[\hat{H}_{\mathrm{TL},c}, \hat{H}_{\mathrm{TL},s}\right] = 0$$

$\Rightarrow$ one version of spin-charge separation

- $\omega_c(q_n) \approx v_c|q_n|, \quad v_c = \sqrt{v_{J,c}v_{N,c}}\,, \quad K_c = \sqrt{v_{J,c}/v_{N,c}}$
- $\omega_s(q_n) \approx v_s|q_n|, \quad v_s = \sqrt{v_{J,s}v_{N,s}}\,, \quad K_s = \sqrt{v_{J,s}/v_{N,s}}$



- also coefficients c_n and s_n for spin and charge

What one can compute from this: the compressibility

- what matters is the charge part of $\hat{H}_{\mathrm{TL}}^{(1/2)}$:

$$\hat{H}_{\mathrm{TL},c} = \sum_{n\neq 0} \omega_c(q_n)\hat{\alpha}^{\dagger}_{n,c}\hat{\alpha}_{n,c} + \frac{\pi}{2L}\left(v_{N,c}\hat{\mathcal{N}}_c^2 + v_{J,c}\hat{\mathcal{J}}_c^2\right)$$

- κ is proportional inverse of second derivative of E_0 with respect to $\left\langle \hat{\mathcal{N}}_c \right\rangle$:

$$\Rightarrow \qquad \kappa \sim \left(\frac{\partial^2 E_0}{\partial \left\langle \hat{\mathcal{N}}_c \right\rangle^2}\right)^{-1}_L$$

$$\Rightarrow \qquad \kappa \sim \frac{1}{v_{N,c}}$$

- temperature $T = 0$ compressibility:

$$\frac{\kappa}{\kappa^0} = \frac{v_F}{v_{N,c}} = K_c\,\frac{v_F}{v_c}$$

What one can compute from this: specific heat

- what matters is the boson part of $\hat{H}_{\mathrm{TL}}^{(1/2)}$:

$$\hat{H}_b = \sum_{n\neq 0} \omega_c(q_n)\hat{\alpha}^\dagger_{n,c}\hat{\alpha}_{n,c} + \sum_{n\neq 0} \omega_s(q_n)\hat{\alpha}^\dagger_{n,s}\hat{\alpha}_{n,s}$$

- two contributions linear in T (non-interacting bosons, Debye):

$$\begin{aligned} \Rightarrow c_V &= \gamma T \\ \frac{\gamma}{\gamma_0} &= \frac{1}{2}\left(\frac{v_F}{v_c} + \frac{v_F}{v_s}\right) \\ \gamma_0 &= 2\,\frac{\pi}{3} \end{aligned}$$

What one can compute from this: the spin susceptibility

- what matters is the spin part of $\hat{H}_{\mathrm{TL}}^{(1/2)}$:

$$\hat{H}_{\mathrm{TL},s} = \sum_{n\neq 0} \omega_s(q_n)\hat{\alpha}^\dagger_{n,s}\hat{\alpha}_{n,s} + \frac{\pi}{2L}\left(v_{N,s}\hat{\mathcal{N}}_s^2 + v_{J,s}\hat{\mathcal{J}}_s^2\right)$$

- add Zeeman term: $\hat{H}_Z = -h\hat{\mathcal{N}}_s = -h\left(\hat{\mathcal{N}}_{+,s} + \hat{\mathcal{N}}_{-,s}\right)$

- minimize ground state energy (no bosons!) with respect to $\left\langle \hat{\mathcal{N}}_s \right\rangle$:

$$\Rightarrow \qquad \frac{\pi}{L} v_{N,s}\left\langle \hat{\mathcal{N}}_s \right\rangle - h = 0$$

$$\Rightarrow \qquad \left\langle \hat{\mathcal{N}}_s \right\rangle \sim \frac{h}{v_{N,s}}$$

$$\Rightarrow \qquad \chi_s = \frac{\partial\left\langle \hat{\mathcal{N}}_s \right\rangle}{\partial h} \sim \frac{1}{v_{N,s}}$$

- temperature $T = 0$ spin susceptibility:

$$\frac{\chi_s}{\chi_s^0} = \frac{v_F}{v_{N,s}} = K_s\,\frac{v_F}{v_s}$$

What one can compute from this: the charge stiffness (Drude weight)

- what matters is the charge part of $\hat{H}^{(1/2)}_{\mathrm{TL}}$:

$$\hat{H}_{\mathrm{TL},c} = \sum_{n\neq 0} \omega_c(q_n)\hat{\alpha}^{\dagger}_{n,c}\hat{\alpha}_{n,c} + \frac{\pi}{2L}\left(v_{N,c}\hat{\mathcal{N}}_c^2 + v_{J,c}\hat{\mathcal{J}}_c^2\right)$$

- add magnetic flux: $\hat{\mathcal{J}}_c \longrightarrow \hat{\mathcal{J}}_c - \Phi/\pi$

- take second derivative of ground state energy with respect to Φ:

$$D \sim v_{J,c}$$

$$\frac{D}{D_0} = \frac{v_c K_c}{v_F}$$

Low-temperature thermodynamics

- q-dependence of boson dispersion (and thus of interaction) kept

- spinless: only two independent parameters, e.g. specific heat can be computed from Drude weight and compressibility

- spinfull: only four independent parameters

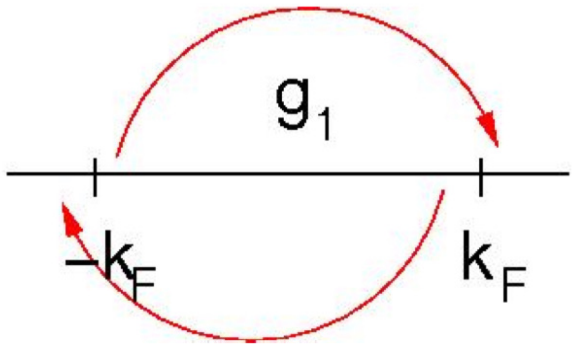
- spinfull: if SU(2) symmetry $\longrightarrow$ $K_s=1$, e.g. specific heat can be computed from Drude weight, compressibility and spin susceptibility

$\longrightarrow$ TL liquid parameters

Add terms to the model

- curvature of fermionic single-particle dispersion

- two-particle backscattering



$$\hat{H}_{g_1} = \sum_{\sigma,\sigma'} \int dx \left(g_{1,\parallel}\, \delta_{\sigma,\sigma'} + g_{1,\perp}\, \delta_{\sigma,-\sigma'} \right) \hat{\psi}^\dagger_{+,\sigma}(x) \hat{\psi}^\dagger_{-,\sigma'}(x) \hat{\psi}_{+,\sigma'}(x) \hat{\psi}_{-,\sigma}(x)$$

- umklapp scattering (lattice)

$\longrightarrow$ as long as system stays gapless all RG irrelevant

(Solyom ´79)

Consider microscopic models

1. barely doable

- perform RG analysis starting from high energies
- determine fixed-point values of $g_2(0)$ and $g_4(0)$ which depend on all parameters
- obtain the low-temperature thermodynamics and its dependence on all parameters

2. often doable

- compute $F(\langle \hat{\mathcal{N}}_c \rangle$ T, h, ϕ) or $E_0(\langle \hat{\mathcal{N}}_c \rangle$, h, ϕ)
- use numerical (DMRG) or analytical (Bethe ansatz) method
- take derivatives and obtain TL liquid parameters
- verify consistency
- if gapless, model is in the TL liquid universality class
- obtain the low-temperature thermodynamics and its dependence on all parameters

$\longrightarrow$ one part of TL liquid universality

Correlation functions of the TL model at low-energies

- contain additional (more interesting) physics
- here: focus on single-particle (2-point) functions
- use exact bosonization of the fermionic field operators
- leads to expressions in x and t

$\uparrow, \downarrow$

$\pm$

$$i G^{>}_{\alpha,s}(x,t) = \langle \psi_{\alpha,s}(x,t) \psi^{\dagger}_{\alpha,s}(0,0) \rangle$$

$$\omega(q_n) = |q_n| v(q_n)$$

$$G^{>}_{\alpha,s}(x,t) = G^{<}_{\alpha,s}(-x,-t)$$

non-linear

finite system, PBC

$$i\, G^{>}_{+}(x,t) = i\, [G^{>}_{+}]^{0}(x,t)\, e^{F(x,t)}$$

$$F(x,t) = \frac{1}{2} \sum_{\nu=c,s} \sum_{n=1}^{\infty} \frac{1}{n} \left[e^{i q_n x} \left(e^{-i\omega_\nu(q_n)t} - e^{-i v_{\mathrm{F}} q_n t} \right) + 2\gamma_\nu(q_n) \left(\cos(q_n x) e^{-i\omega_\nu(q_n)t} - 1 \right) \right]$$

$$\gamma_\nu(q) = [K_\nu(q) + 1/K_\nu(q) - 2]/4$$

depends on q

$$[G^{>}_{+}]^{0}(x,t) = \frac{e^{i(k_{\mathrm{F}}+\pi/L)x}}{L} \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{n} e^{i q_n (x - v_{\mathrm{F}} t + i0^{+})} \right\} \overset{L \to \infty}{=} \frac{1}{2\pi} \frac{e^{i k_{\mathrm{F}} x}}{x - v_{\mathrm{F}} t + i0^{+}}$$

large x asymptotics?

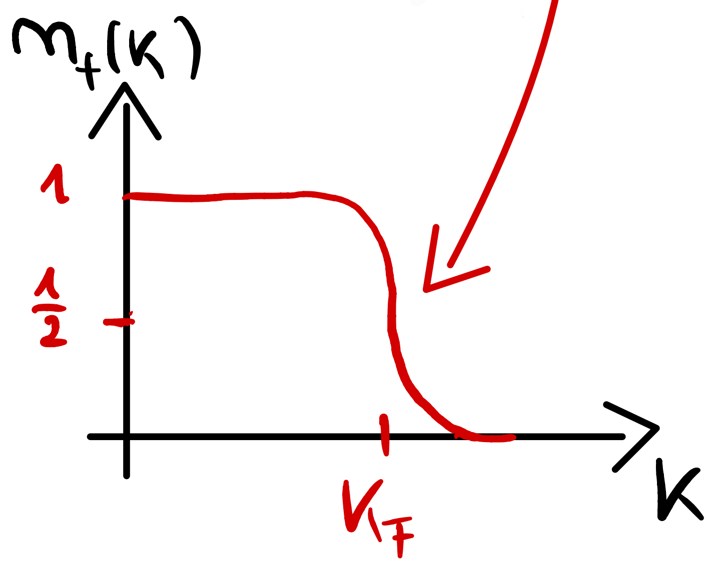
Momentum distribution function TL model

$$n_+(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} iG_+^<(x,0)$$

use asymptotic analysis to show

$$\frac{1}{2} - n_+(k) \sim \begin{cases} |k-k_F|^\alpha \operatorname{sign}(k-k_F) & \text{for } 0<\alpha<1 \\ (k-k_F)\ln|k-k_F| & \text{for } \alpha=1 \\ (k-k_F) & \text{for } \alpha>1 \end{cases}$$

$$\alpha = \gamma_c(0) + \gamma_s(0)$$



to show this it is not required to replace

$$\gamma_\nu(q) \to \gamma_\nu(0)$$

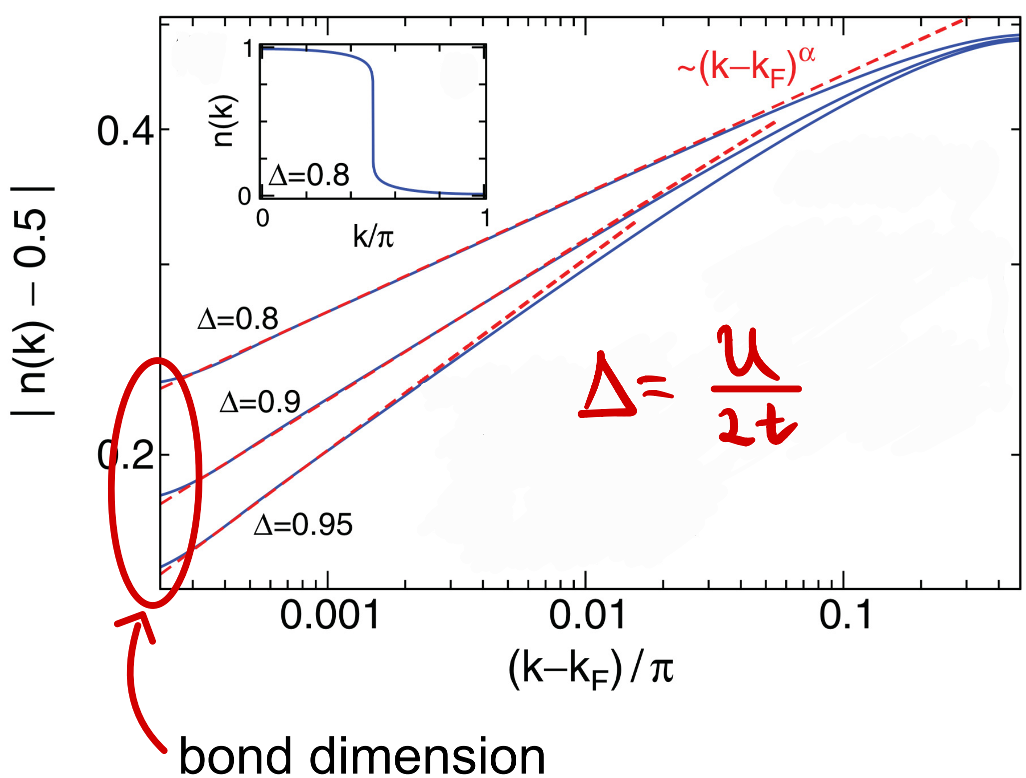
and regularize the q-integral in F(x,0) by hand
(still often done!)

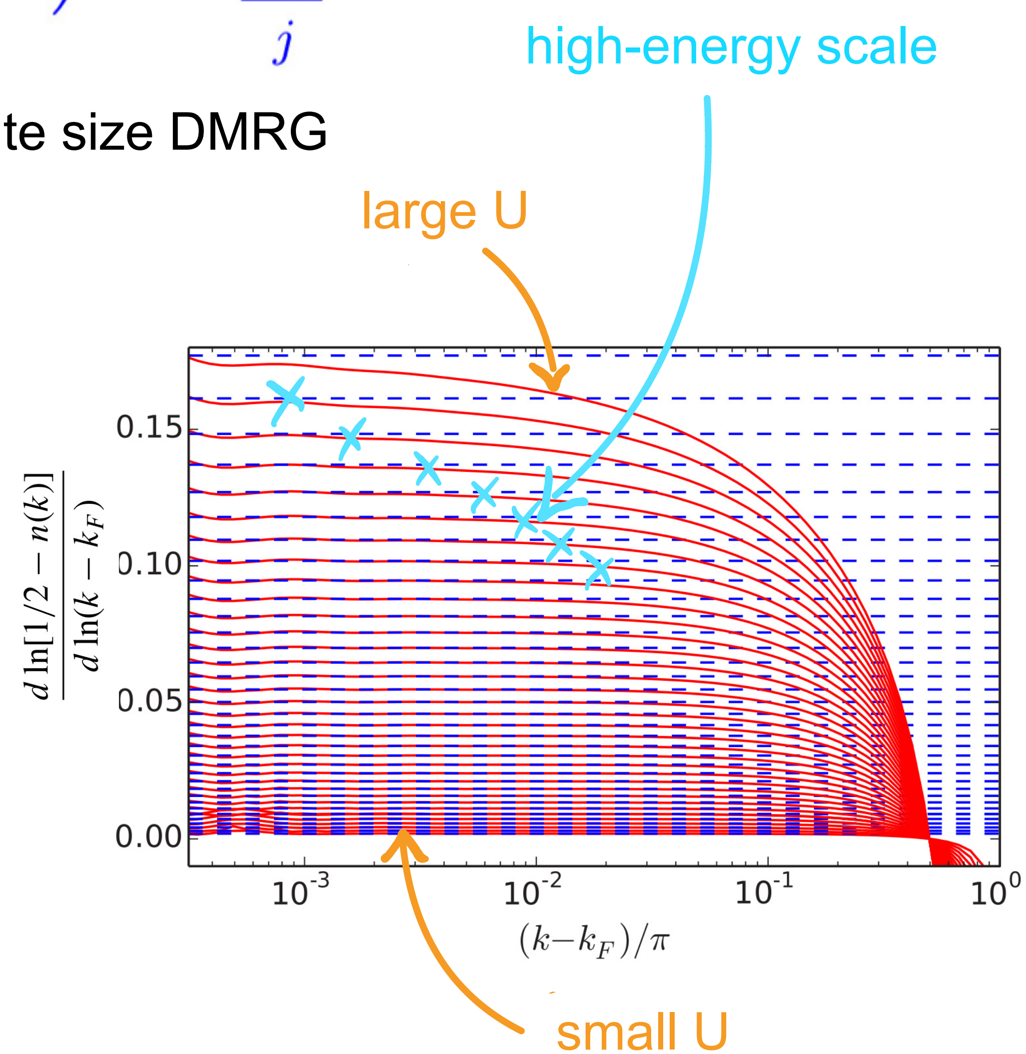
(Theumann \`67, V.M. \`99)

Can this be verified in a microscopic lattice model?

$$\hat{H} = -t\sum_j \left(\hat{c}_j^\dagger \hat{c}_{j+1} + \text{H.c.}\right) + U\sum_j \hat{n}_j \hat{n}_{j+1}$$

half filling, gapless for -2<U/t<2, use infinite size DMRG





(Karrasch, Moore `12)

(Kennes,…, V.M. `14)

Even more interesting physics: single-particle spectral function (TL model)

$$\rho_+^>(k_n,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\int_{-L/2}^{L/2} dx\, e^{-ik_n x}\; iG_+^>(x,t)$$

$$i\,G_+^>(x,t) = i\,[G_+^>]^0(x,t)\,e^{F(x,t)}$$

$$F(x,t) = \frac{1}{2}\sum_{\nu=c,s}\sum_{n=1}^{\infty}\frac{1}{n}\Big[e^{iq_n x}\big(e^{-i\omega_\nu(q_n)t} - e^{-iv_{\mathrm{F}}q_n t}\big) + 2\gamma_\nu(q_n)\big(\cos(q_n x)e^{-i\omega_\nu(q_n)t} - 1\big)\Big]$$

$\longrightarrow$ directly perform q, t, x integrals numerically? $\lightning$

$\longrightarrow$ momentum dependence of potential such that integrals doable analytically? $\lightning$

$\longrightarrow$ approximations such that (at least) F(x,t) explicit? $\checkmark$

Approximated Green functions

$$i\, G_{+}^{>}(x,t) = i\, [G_{+}^{>}]^{0}(x,t)\, e^{F(x,t)}$$

$$F(x,t) = \frac{1}{2} \sum_{\nu=c,s} \sum_{n=1}^{\infty} \frac{1}{n} \left[e^{i q_n x} \left(e^{-i\omega_\nu(q_n)t} \cancel{- e^{-i v_{\mathrm{F}} q_n t}} \right) + 2\gamma_\nu(q_n) \left(\cos(q_n x) e^{-i\omega_\nu(q_n)t} - 1 \right) \right]$$

$$[G_{+}^{>}]^{0}(x,t) = \frac{e^{i(k_{\mathrm{F}}+\pi/L)x}}{L} \exp\left\{ \cancel{\sum_{n=1}^{\infty} \frac{1}{n} e^{i q_n (x - v_{\mathrm{F}} t + i0^{+})}} \right\}$$

$\longrightarrow$ rest is UV divergent

linearize and regularize by hand in the UV—„hope“ that this does not affect the IR

Approximated Green functions

$$\boxed{\times \exp(-q_n 0^+)}$$

$$\boxed{= \gamma_\nu \exp(-q_n \Lambda)}$$

$$i\,G^>_+(x,t) = i\,[G^>_+]^0(x,t)\,e^{F(x,t)}$$

$$F(x,t) = \frac{1}{2} \sum_{\nu=c,s} \sum_{n=1}^{\infty} \frac{1}{n} \Big[e^{iq_n x} \big(e^{-i\omega_\nu(q_n)t} - \cancel{e^{-iv_\mathrm{F} q_n t}} \big) + 2\gamma_\nu(q_n) \big(\cos(q_n x) e^{-i\omega_\nu(q_n)t} - 1 \big) \Big]$$

$$[G^>_+]^0(x,t) = \frac{e^{i(k_\mathrm{F}+\pi/L)x}}{L} \exp\left\{ \cancel{\sum_{n=1}^{\infty} \frac{1}{n} e^{iq_n(x - v_\mathrm{F} t + i0^+)}} \right\}$$

$$\boxed{\omega_\nu(q) \rightarrow v_\nu q}$$

$$[G^>_+]_{(\mathrm{i})}(x,t) = \frac{e^{ik_\mathrm{F}x}}{2\pi} \prod_{\nu=c,s} \left[\frac{1}{x - v_\nu t + i0^+} \right]^{1/2}$$

$$\times \left[\frac{\Lambda^2}{(x - v_\nu t + i\Lambda)(x + v_\nu t - i\Lambda)} \right]^{\gamma_\nu/2}$$

(done by many people, e.g., Luther, Peschel `74, V.M., Schönhammer `92, Voit `93)

Approximated Green functions

$$\boxed{\times \exp\left(-q_n \Lambda\right)}$$

$$i\, G_+^>(x,t) = i\, [G_+^>]^0(x,t)\, e^{F(x,t)}$$

$$F(x,t) = \frac{1}{2} \sum_{\nu=c,s} \sum_{n=1}^{\infty} \frac{1}{n} \Big[e^{i q_n x} \Big(e^{-i\omega_\nu(q_n)t} \cancel{- e^{-i v_F q_n t}} \Big) + 2\gamma_\nu \cancel{(q_n)} \Big(\cos(q_n x) e^{-i\omega_\nu(q_n)t} - 1 \Big) \Big]$$

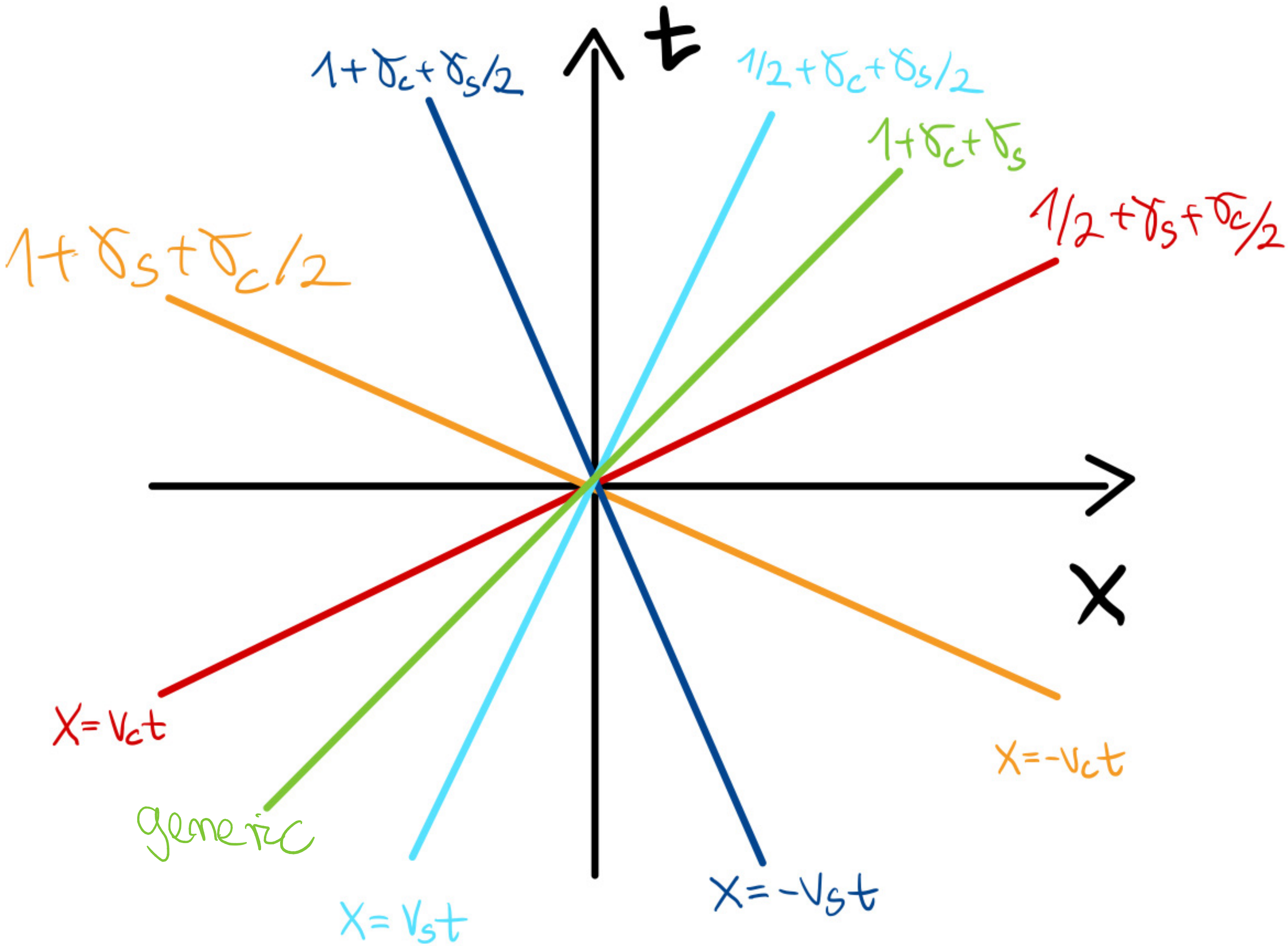
$$[G_+^>]^0(x,t) = \frac{e^{i(k_F + \pi/L)x}}{L} \exp\left\{ \cancel{\sum_{n=1}^{\infty} \frac{1}{n} e^{i q_n (x - v_F t + i0^+)}} \right\}$$

$$\boxed{\omega_\nu(q) \rightarrow v_\nu q}$$

$$[G_+^>]_{(\mathrm{ii})}(x,t) = \frac{e^{i k_F x}}{2\pi} \prod_{\nu=c,s} \left[\frac{1}{x - v_\nu t + i\Lambda} \right]^{1/2}$$

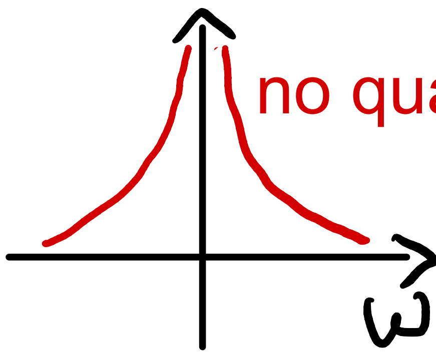
$$\times \left[\frac{\Lambda^2}{(x - v_\nu t + i\Lambda)(x + v_\nu t - i\Lambda)} \right]^{\gamma_\nu/2}$$

What they have in common: decay in x-t plane

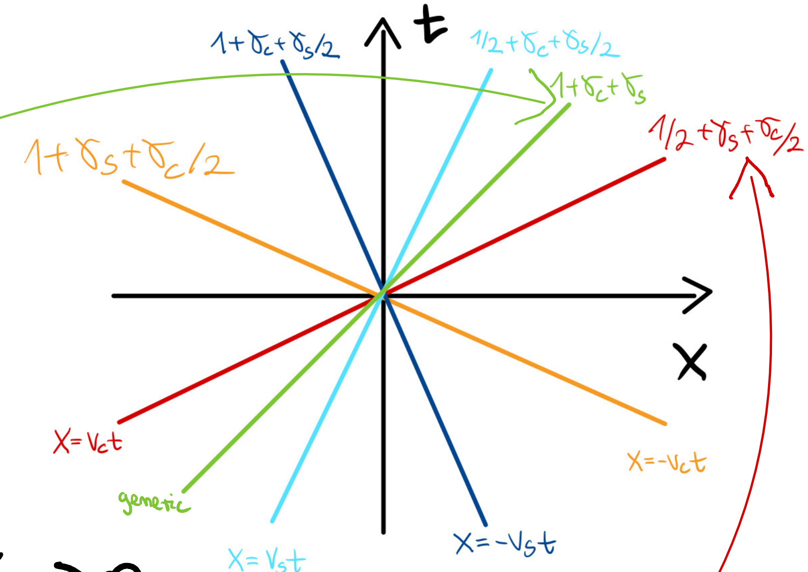


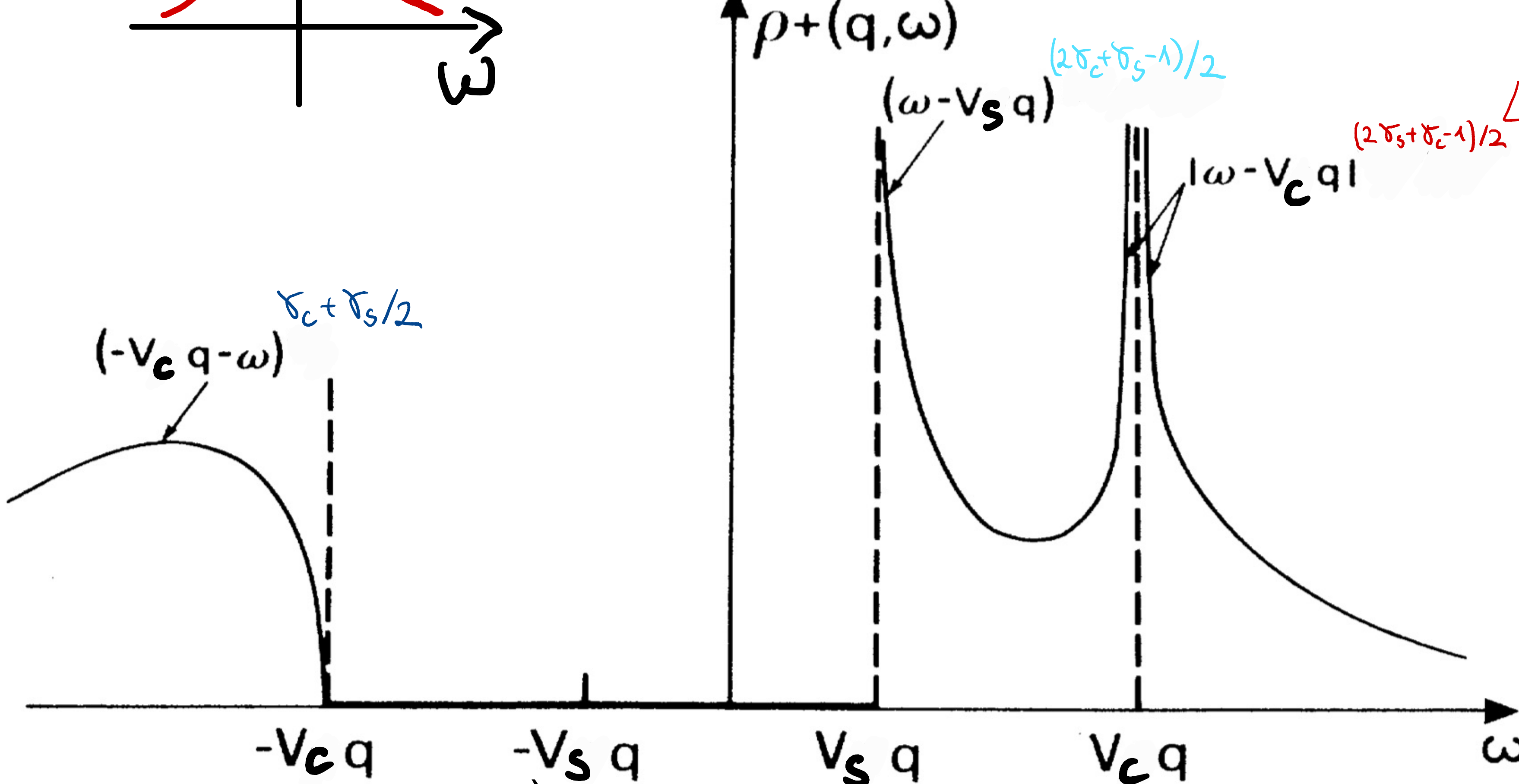
What they have in common

$$S_t(\overset{k_F}{0},\omega) \sim |\omega|^{\overbrace{\gamma_c+\gamma_s-1}^{\alpha}}$$



no quasi-particle





(V.M., Schönhammer `92, Voit `93)

But…

take momentum dependence serious (no ad hoc regularization and linearization)

$$i\,G^{>}_{+}(x,t) = i\,[G^{>}_{+}]^{0}(x,t)\,e^{F(x,t)}$$

$$F(x,t) = \frac{1}{2}\sum_{\nu=c,s}\sum_{n=1}^{\infty}\frac{1}{n}\Big[e^{iq_n x}\big(e^{-i\omega_\nu(q_n)t} - e^{-iv_F q_n t}\big) + 2\gamma_\nu(q_n)\big(\cos(q_n x)e^{-i\omega_\nu(q_n)t} - 1\big)\Big]$$

arbitrary

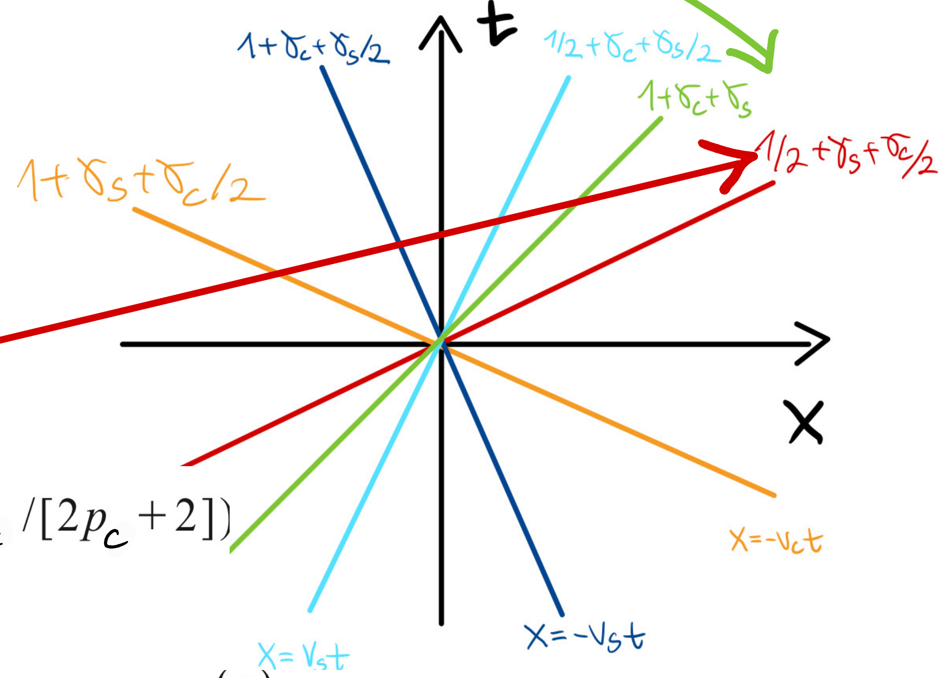
$$\tilde{G}^{<}_{+}(s,s') \equiv G^{<}_{+}(x[s,s'],t[s,s']) \qquad s = x - ct \qquad s' = x + ct$$

$$c \neq v_c(0),\; v_s(0)$$

$$\tilde{G}^{<}_{+}(s,s') \sim s'^{\,-(1+\alpha)}$$

$$c = v_c(0)$$

$$\tilde{G}^{<}_{+}(s,s') \sim s'^{\,-(1/2+\gamma_s+\gamma_c/2+1/[2p_c+2]+\gamma_c/[2p_c+2])}$$

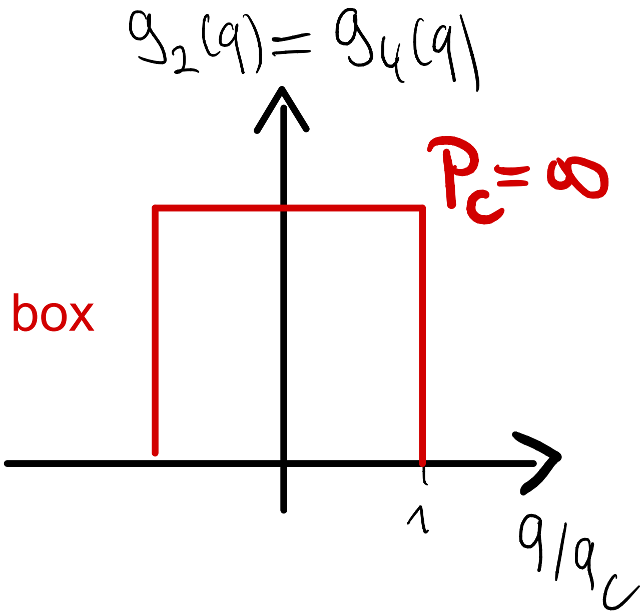


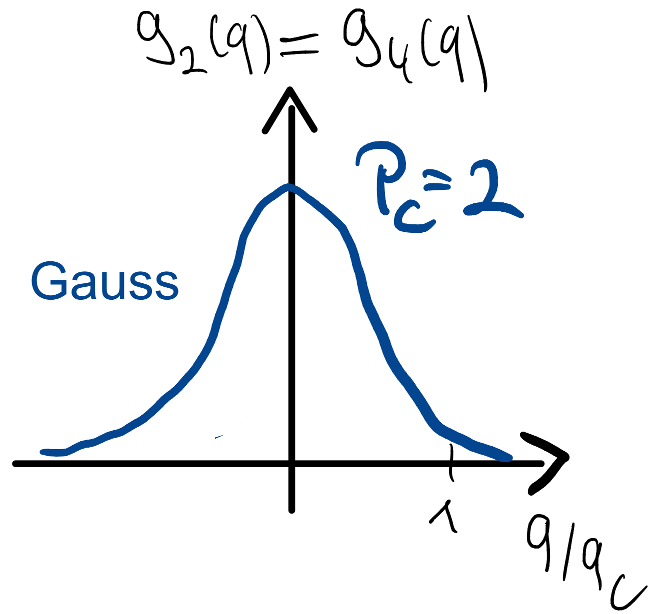
p_c: smallest number $n \in \mathbb{N}\cup\{\infty\}$ with $[v_c]^{(n)}(0) \neq 0$

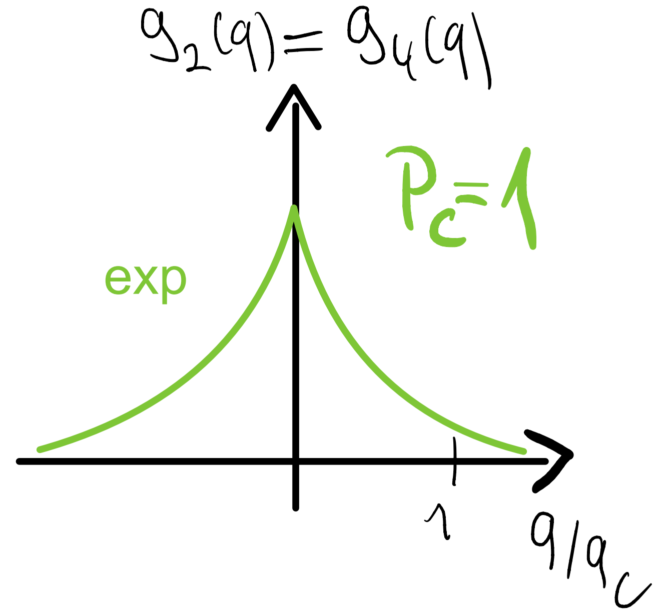
(V.M. \`99)

But…

take momentum dependence serious (no ad hoc regularization and linearization)







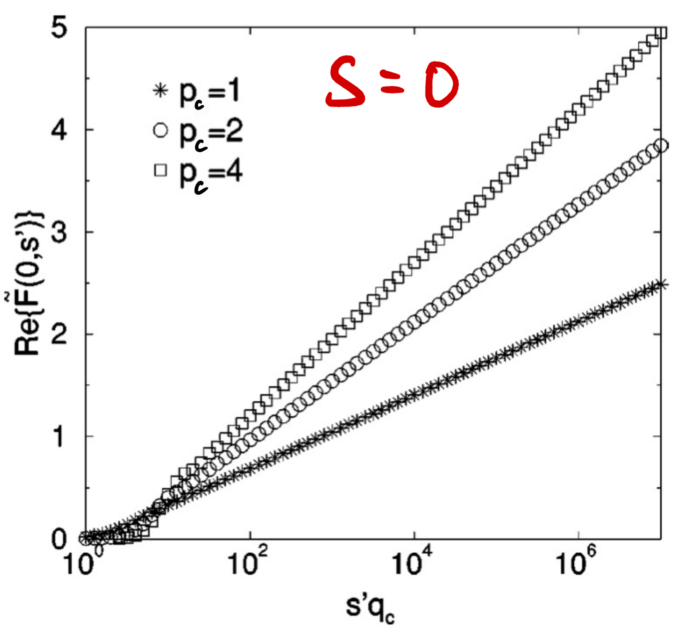


FIG. 1. Re$\{\tilde{F}(0,s')\}$ as a function of $s'q_c$ for different p_c .

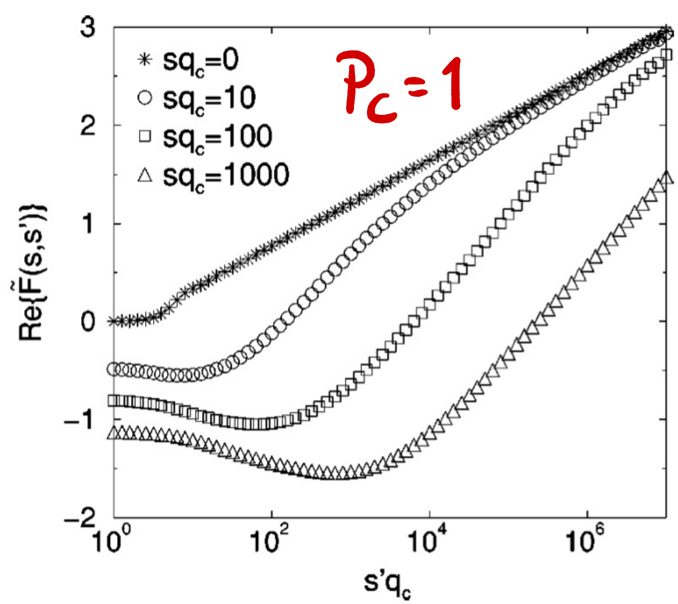


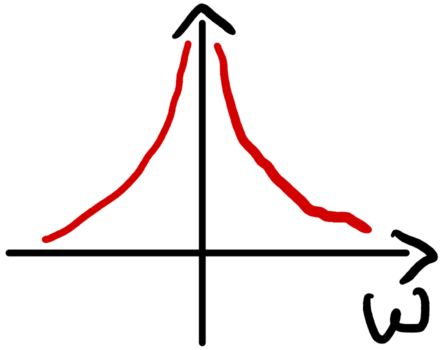
FIG. 2. Re$\{\tilde{F}(s,s')\}$ as a function of $s'q_c$ for different sq_c .

(V.M. \`99)

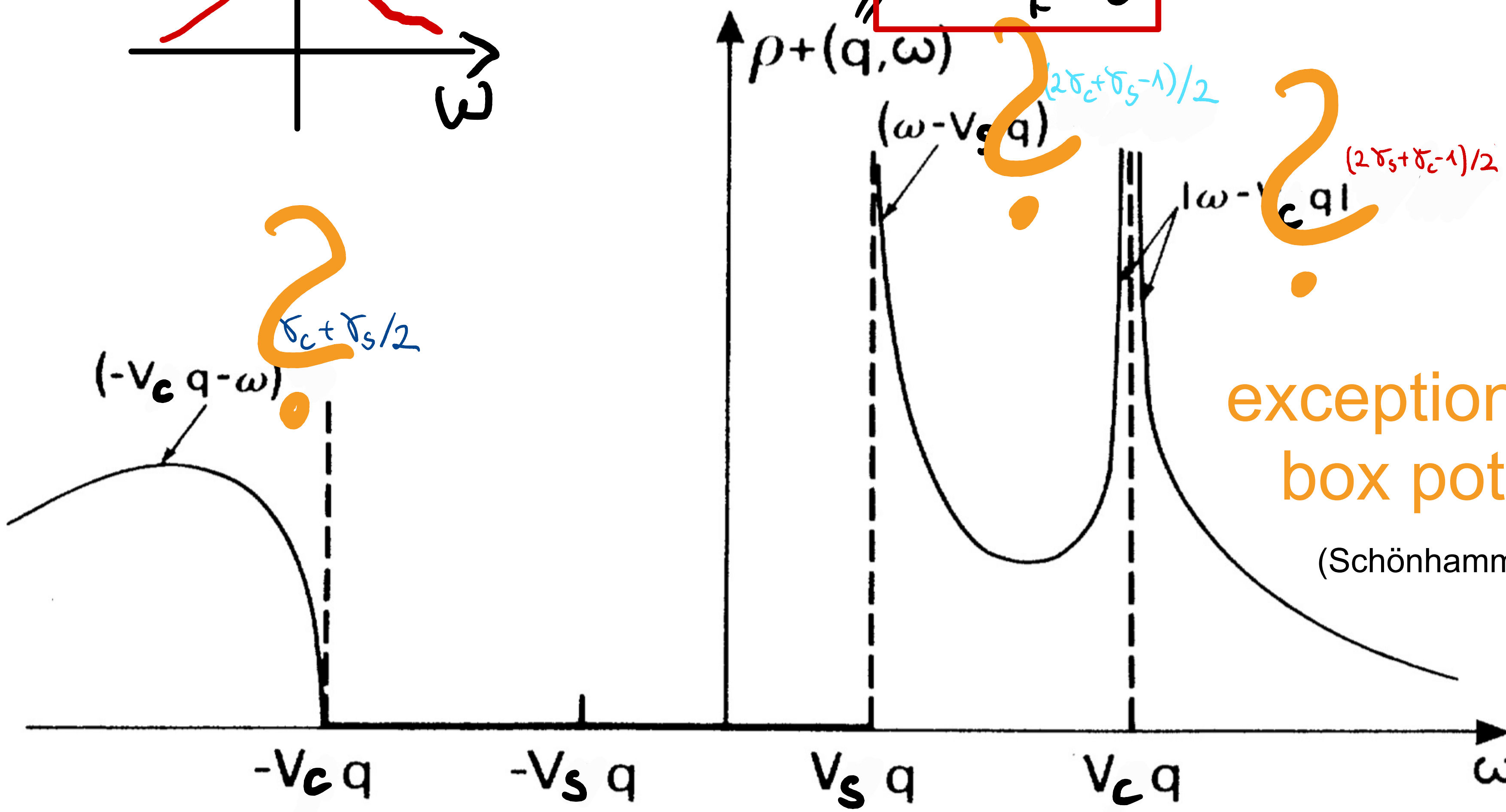
What happens to the spectral function?

$$\rho_+(k_F,\omega) \sim |\omega|^{\overbrace{\gamma_c+\gamma_s-1}^{\alpha}}$$

all energy scales to 0



$\boxed{k-k_F>0}$ finite energy scale



exception:
box potential

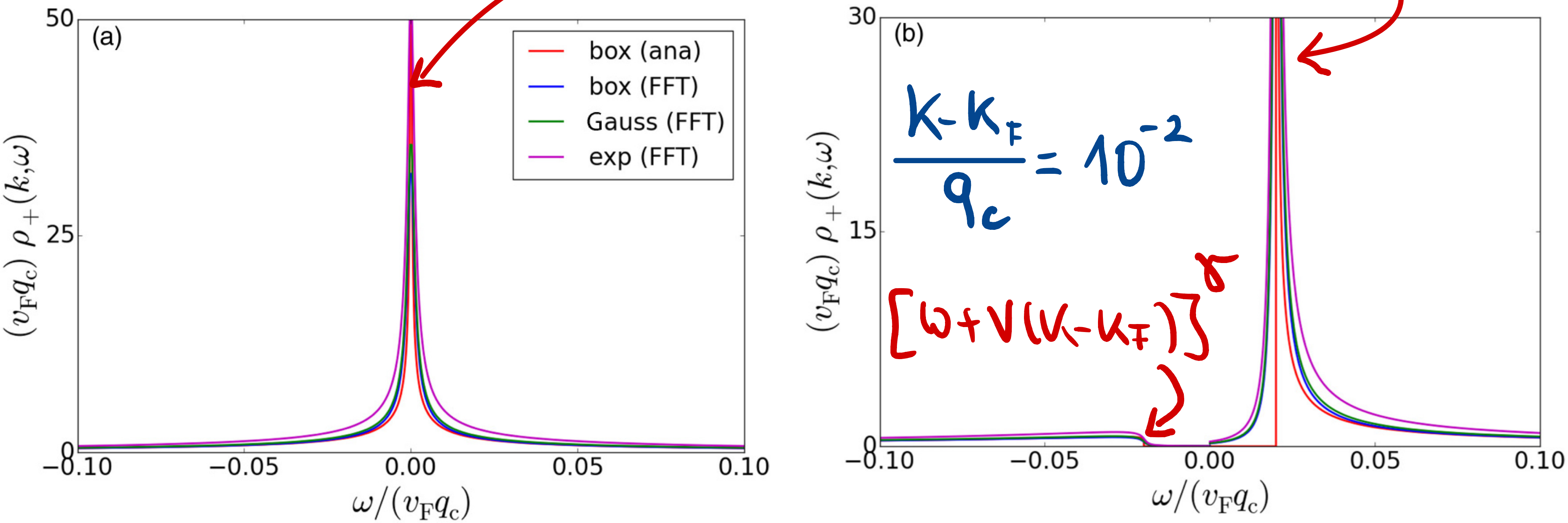
(Schönhammer, V.M. `93)

Exact spectral function with momentum dependent potential

- recursive method for large but finite L
- use FFT for time Fourier transform
- requires multiplication by $\exp\{-\chi|t|\}$ $\longrightarrow$ broadening of delta peaks
- consider spinless case

$|\omega|^{2\gamma-1}$

$[\omega - v(k-k_F)]^{\gamma-1}$

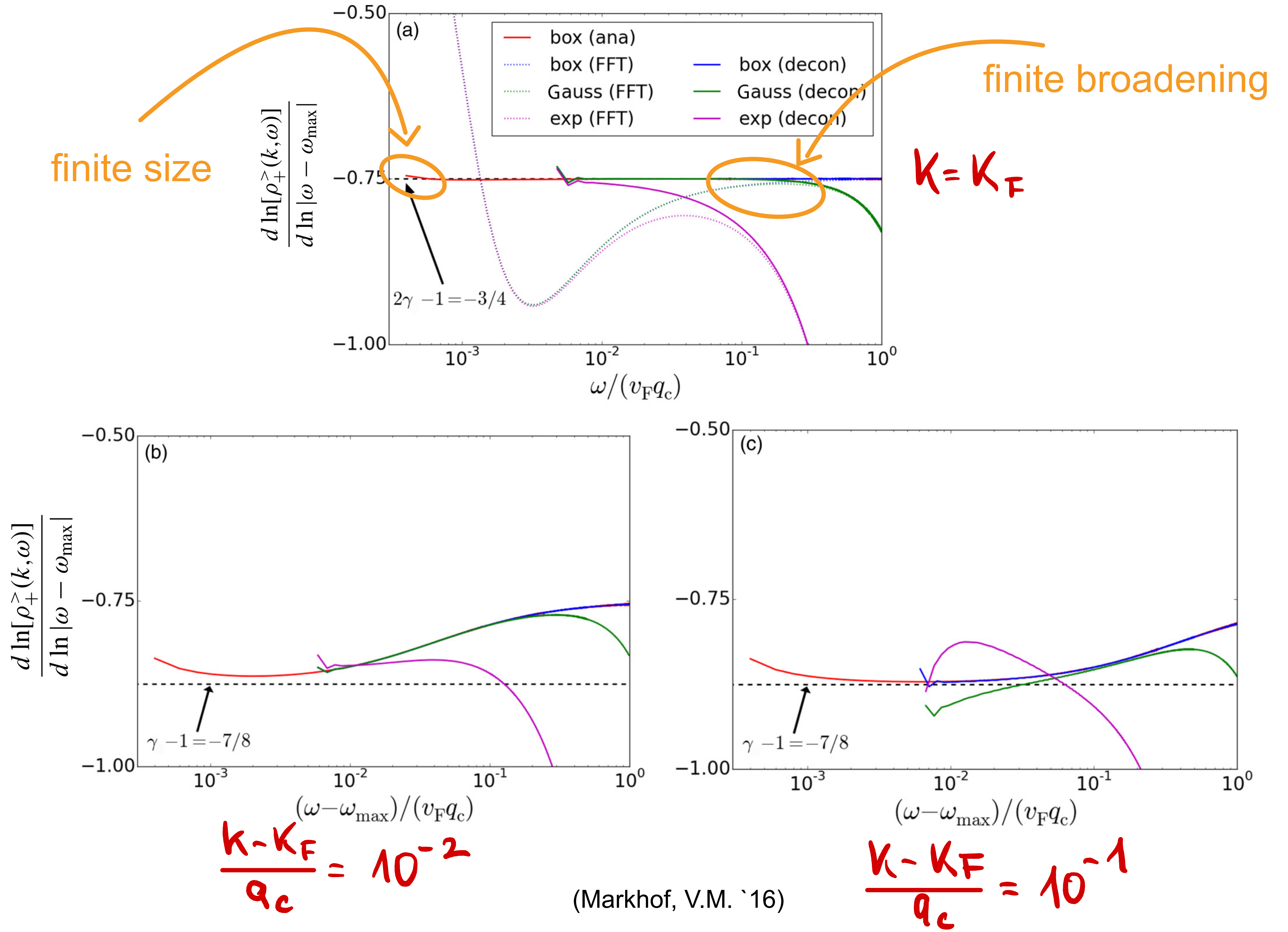


$$\frac{k-k_F}{q_c} = 10^{-2}$$

$[\omega + v(k-k_F)]^{\gamma}$

$$\frac{\chi}{v_F q_c} = 10^{-3}, \quad \gamma(0) = \frac{1}{8}, \quad v = 2v_F, \quad m_c = 2\times 10^4$$

Exact spectral function with momentum dependent potential



(Markhof, V.M. `16)

Conclusion

- there is no indication that the spectral function of the TL model shows power-law behavior for $k-k_F \neq 0$ and generic two-particle interactions

- as the TL model is the low-energy fixed point model: $k-k_F \neq 0$ power-laws are not part of TL liquid theory

- microscopic (integrable) models: if power laws other reason(s)

- be aware how difficult it is to „prove" power-law behavior from numerical data

- non-linear TL liquid phenomenology: does not properly deal with the UV (ad hoc regularization) (Imambekov et al. ´12)

Excitations in 1d — non-interacting electrons

reminder of your statistical mechanics/solid state physics class

- low-temperature ($T/\epsilon_F \ll 1$) specific heat of electrons (Sommerfeld)

$$C_V \approx \frac{\pi^2}{3} T\, g(\epsilon_F) = 2\,\frac{\pi}{3}\frac{T}{v_F}\,, \quad k_B = 1$$

- specific heat of phonons in d dimensions (Debye)

$$C_V = \frac{d}{dT}\sum_b \int \frac{d^d k}{(2\pi)^d}\frac{\omega_b(k)}{e^{\omega_b(k)/T} - 1}$$

low-energy and $d = 1$, single branch: $\omega(k) = c_s|k|$

$$\begin{aligned} C_V &\approx \frac{d}{dT} 2\int_0^\infty \frac{dk}{2\pi}\frac{c_s k}{e^{c_s k/T} - 1} \\ &= \frac{d}{dT}\frac{c_s}{\pi}\left(\frac{T}{c_s}\right)^2 \int_0^\infty dx \frac{x}{e^x - 1} \\ &= \frac{\pi}{3}\frac{T}{c_s} \end{aligned}$$

- same spectrum and degeneracy of low-energy excited states!?

(Schönhammer & V.M., Am. J. Phys. **64**, 1168 (1996))

Approximated Green functions

$$\boxed{\times \exp\left(-q_n \Lambda\right)}$$

$$i\, G_+^>(x,t) = i\,[G_+^>]^0(x,t)\, e^{F(x,t)}$$

$$F(x,t) = \frac{1}{2} \sum_{\nu=c,s} \sum_{n=1}^{\infty} \frac{1}{n}\left[e^{i q_n x}\left(e^{-i\omega_\nu(q_n)t} - e^{-i v_F q_n t}\right) + 2\gamma_\nu(\cancel{q_n})\left(\cos(q_n x) e^{-i\omega_\nu(q_n)t} - 1\right)\right]$$

$$[G_+^>]^0(x,t) = \frac{e^{i(k_F+\pi/L)x}}{L} \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} e^{i q_n (x - v_F t + i0^+)}\right\}$$

$$\boxed{\omega_\nu(q) \to v_\nu q}$$

$$[G_+^>]_{(\mathrm{iii})}(x,t) = \frac{e^{i k_F x}}{2\pi} \frac{1}{x - v_F t + i0^+} \prod_{\nu=c,s} \left[\frac{x - v_F t + i\Lambda}{x - v_\nu t + i\Lambda}\right]^{1/2}$$

$$\times \left[\frac{\Lambda^2}{(x - v_\nu t + i\Lambda)(x + v_\nu t - i\Lambda)}\right]^{\gamma_\nu/2}$$

purely fermionic

(Dzyaloshinskii, Larkin \`73, Voit \`93, Maebashi, Takada \`14)