Calogero models and the non-abelian quantum Hall effect

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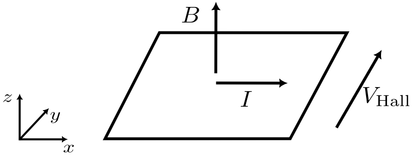
[Based on [JEB-Matsuo 2401.03087]]

Hall effect

- The **classical Hall effect** has been discovered by Edwin Hall in 1879. It is the observation of a transverse voltage for a conductor in a magnetic field. It is due to the Lorentz force acting on the electrons.



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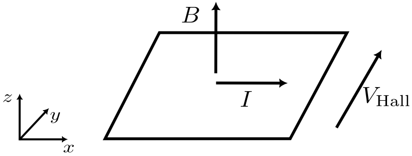
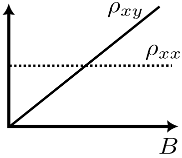


- As a result, the resistivity is expected to behave as follows

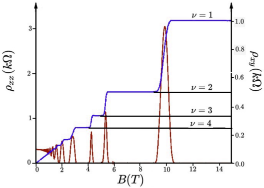


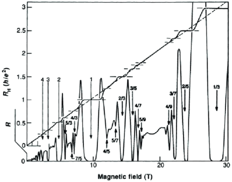
Quantum Hall effect

• The **quantum Hall effect** (QHE) refers to the observation of plateaux in the resistivity at low temperatures and strong magnetic field. The measured resistance is given by the formula

$$R_{xy} = \frac{V_{\text{Hall}}}{I} = \frac{h}{e^2 \nu}.$$

⇝ Plateaux are labeled by the **filling factor** ν, it takes either integer or fractional values.





⇝ Integer QHE observed by Klaus von Klitzing in 1980 in MOSFET.

⇝ Fractional QHE discovered by Daniel Tsui and Horst Störmer in 1982 in GaAs samples.

Integer quantum Hall effect

- The **integer** QHE has a simple quantum mechanical explanation. It is due to the **impurities** in the material that lift the macroscopic degeneracy of the Landau levels. Impurities also turn extended states into localized states. Since these states cannot carry a current, they do not contribute to the conductivity.

$\Rightarrow$ Observe plateaux of length proportional to the amount of impurities!

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• The filling factor ν takes a specific set of rational values. The first explanation of the FQHE was proposed by Laughlin in 1983. He wrote the expression of an effective ground state wave function for the values filling factors $\nu = 1/k$ with k odd,

$$\psi(\boldsymbol{x}) = \prod_{a<b} (x_a - x_b)^k e^{-\frac{eB}{4\hbar} \sum_a |x_a|^2}.$$

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• The model exhibits excitations called **quasi-holes** with the wave function

$$\psi(\zeta_i, \boldsymbol{x}) = \prod_{i<j} (\zeta_i - \zeta_j)^{1/k} \times \prod_i \prod_a (\zeta_i - x_a) \times \prod_{a<b} (x_a - x_b)^k e^{-\frac{eB}{4\hbar} \sum_a |x_a|^2},$$

They carry a fraction of the charge of the electrons. They also obey *a fractional statistics*, namely they are **(abelian) anyons**!

- Model of the fractional QHE with more general filling factors $\nu = p/(k + pn)$ involve **non-abelian anyons**. The statistics of these anyons (braiding) is given in terms of a non-trivial matrix transformation.

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• In fact, anyons are **topologically protected quantum states**. They are very good candidates to realize fault-tolerant quantum processors!

• So far, there have been no confirmed observation of non-abelian anyons in condensed matter systems, but they have been realized on quantum processors **[G. Q. AI and Collaborators, Nature 618 (2023) 264–269]**. Some cold atom systems are expected to also show similar properties.

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  - Field theory (IR) description using 3d Chern-Simons theory

    $\rightsquigarrow$ Abelian FQHE: $U(1)$ Chern-Simons level k ($\nu = 1/k$)

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$\Rightarrow$ So, what's new?

- In **[2401.03087]**, we introduce **a new quantum model** for the non-abelian FQHE. It is obtained as the diagonalization of a matrix model proposed by **[Dorey, Tong, Turner 2016]**. This model describes the vortices of 3d $U(p)$ Chern-Simons theory.

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$\rightsquigarrow$ **It relates the Chern-Simons and CFT approaches!**

• The Hamiltonian of our quantum system is a higher spin version of the Calogero Hamiltonian, it has been shown to be an **integrable system**.

$\rightsquigarrow$ **Use the algebraic techniques of integrable systems to do exact calculations!**

Outline

2. Spin Calogero model

Definition of the model

- The model depends of the following parameters:
  - $N \in \mathbb{Z}^{\geq 0}$ the number of vortices in Chern-Simons theory
  - $p \in \mathbb{Z}^{>0}$ the rank of the non-abelian $U(p)$ symmetry
  - $k \in \mathbb{Z}^{\geq 0}$ the level of the Chern-Simons theory (filling factor $\nu = p/(k+p)$).
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- The model describes N particles of coordinates $x_a \in \mathbb{R}$ carrying a representation of the $U(p)$ symmetry. The spin structure is introduced using the bosonic oscillators $\varphi_{i,a}^\dagger$, $\varphi_{i,a}$,

$$[\varphi_{i,a}, \varphi_{j,b}^\dagger] = \delta_{i,j}\delta_{a,b}, \quad i,j = 1\cdots p, \quad a,b = 1\cdots N.$$

$\rightsquigarrow$ At level k, each particle carries a state built from k oscillators $\varphi_{i_1,a}^\dagger \cdots \varphi_{i_k,a}^\dagger |\emptyset\rangle$.

$\Rightarrow$ **The model involves higher order symmetric representations of the $U(p)$ symmetry!**

- The Hamiltonian of the model reads

$$\mathcal{H} = \sum_{a=1}^{N} \left(-\frac{\partial^2}{\partial x_a^2} + B^2 x_a^2 \right) + 2 \sum_{\substack{a,b=1 \\ a<b}}^{N} \frac{J_{a,b} J_{b,a}}{(x_a - x_b)^2},$$

where $J_{a,b}$ are the generators of $\mathfrak{gl}(N)$ acting on the spin components,

$$J_{a,b} = \sum_{i=1}^{p} \varphi_{i,a}^{\dagger} \varphi_{i,b}, \quad [J_{a,b}, J_{c,d}] = \delta_{a,d} J_{c,b} - \delta_{b,c} J_{a,d}.$$

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$$K_{i,j} = \sum_{a=1}^{N} \varphi_{i,a}^{\dagger} \varphi_{j,a}, \quad [K_{i,j}, K_{k,l}] = \delta_{j,k} K_{i,l} - \delta_{i,l} K_{k,j}.$$

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$\Rightarrow$ Let's examine some specific values of p and k!

Specializations

- For $p = 1$, we recover the Calogero Hamiltonian describing the abelian model

$$\mathcal{H}^{(p=1)} = \sum_{a=1}^{N} \left(-\frac{\partial^2}{\partial x_a^2} + B^2 x_a^2 \right) + 2 \sum_{\substack{a,b=1 \\ a<b}}^{N} \frac{k(k+1)}{(x_a - x_b)^2}.$$

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$\Rightarrow$ Let's examine the main properties of this model!

Simplification

- To simplify our analysis, we also perform the following technical manipulations:
  - Conjugation of the Hamiltonian,

$$\tilde{\mathcal{H}} = e^{\frac{B}{2}\sum_a x_a^2}\mathcal{H}e^{-\frac{B}{2}\sum_a x_a^2} = \sum_a \left(-\frac{\partial^2}{\partial x_a^2} + 2Bx_a\frac{\partial}{\partial x_a}\right) + NB + \sum_{a\neq b}\frac{J_{a,b}J_{b,a}}{(x_a-x_b)^2}.$$

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  - Rescaling of the coordinates $x_a \to B^{-1/2}x_a$ and energies $\tilde{\mathcal{H}} \to B^{-1}\tilde{\mathcal{H}}$ to set $B = 1$.
  - Use a polynomial representation for the spin states,

$$\varphi^\dagger_{i_1,a}\cdots\varphi^\dagger_{i_k,a}\,|\emptyset\rangle \to y_{i_1,a}\cdots y_{i_k,a}, \quad \varphi_{i,a} \to \partial_{i,a} = \frac{\partial}{\partial y_{i,a}}.$$

  $\rightsquigarrow$ **States are expressed as polynomial wave functions $\Psi(\boldsymbol{x},\boldsymbol{y})$.**

  They are homogeneous of total degree k in $y_{i,a}$ for each particle a.

Spectrum and eigenfunctions

- To express the wave functions, we first define the wedge states as $N \times N$ determinants,

$$\mathcal{Y}_{\boldsymbol{r}}(\boldsymbol{x}, \boldsymbol{y}) = [y_{i_1} x^{n_1} \wedge \cdots \wedge y_{i_N} x^{n_N}] = \begin{vmatrix} y_{i_1,1}(x_1)^{n_1} & \cdots & y_{i_N,1}(x_1)^{n_N} \\ \vdots & & \vdots \\ y_{i_1,N}(x_N)^{n_1} & \cdots & y_{i_N,N}(x_N)^{n_N} \end{vmatrix}.$$

They are labeled by N-tuple integers $\boldsymbol{r} = (r_1, \cdots, r_N)$, ordered as $0 \leq r_1 < r_2 < \cdots < r_N$, which decompose as $r_a = n_a p + i_a - 1$ with $i_a \in [\![1, p]\!]$ under Euclidean division.

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- For $k \in \mathbb{Z}^{>0}$ fixed, we introduce the following set of wave functions

$$\psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y}) = \Delta(\boldsymbol{x}) \prod_{\alpha=1}^{k} \mathcal{Y}_{\boldsymbol{r}^{(\alpha)}}(\boldsymbol{x},\boldsymbol{y}), \quad \Delta(\boldsymbol{x}) = \prod_{a<b} (x_a - x_b).$$

They are labeled by k tuples of p integers (ordered as before), $\mathfrak{r} = \left\{\boldsymbol{r}^{(1)}, \cdots, \boldsymbol{r}^{(k)}\right\}$. We also assume that re-ordering of $\boldsymbol{r}^{(\alpha)}$ in $\mathfrak{r}$ does not give a new element.

Main result: We proved that the Hamiltonian $\mathcal{H}$ has a triangular action on functions $\psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y})$,

$$\tilde{\mathcal{H}}\psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y}) = E(\mathfrak{r})\psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y}) + \sum_{|\mathfrak{s}|=|\mathfrak{r}|-2} C(\mathfrak{r},\mathfrak{s})\psi_{\mathfrak{s}}(\boldsymbol{x},\boldsymbol{y}),$$

where $C(\mathfrak{r},\mathfrak{s})$ is a coefficient. This can be inverted to write down the eigenfunctions,

$$\Psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y}) = \psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y}) + \sum_{|\mathfrak{s}|\leq|\mathfrak{r}|-2} D(\mathfrak{r},\mathfrak{s})\psi_{\mathfrak{s}}(\boldsymbol{x},\boldsymbol{y})\,.$$

$\rightsquigarrow$ We deduce the energy spectrum

$$E(\mathfrak{r}) = 2|\mathfrak{r}| + N, \quad |\mathfrak{r}| = \sum_{\alpha=1}^{k}\sum_{a=1}^{N}\lfloor r_a^{(\alpha)}/p\rfloor = \sum_{\alpha=1}^{k}\sum_{a=1}^{N} n_a^{(\alpha)}.$$

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⚠ **These wave functions are not linearly independent!**

$\rightsquigarrow$ This can be seen e.g. from the existence of Plücker relations between determinants.

We deduce the ground states wave function from the triangular action:

- When $p|N$: When $k=1$, the ground state is unique and corresponds to $\boldsymbol{r}_0=(0,1,2,\cdots,N-1)$,

$$\mathcal{Y}_{\boldsymbol{r}_0}(\boldsymbol{x},\boldsymbol{y})=[y_1\wedge\cdots\wedge y_p\wedge y_1x\wedge\cdots\wedge y_px\wedge\cdots\wedge y_1x^{m-1}\wedge\cdots\wedge y_px^{m-1}].$$

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- When $N = mp+q$ with $0<q\leq p-1$: The ground state for $k=1$ is $\binom{p}{q}$-fold degenerate. It corresponds to a choice of spin for q extra particles,

$$\mathcal{Y}_{\boldsymbol{r}_0}(\boldsymbol{x},\boldsymbol{y}) = [y_1 \wedge \cdots \wedge y_p \wedge y_1 x \wedge \cdots \wedge y_p x \wedge \cdots \wedge y_1 x^{m-1} \wedge \cdots \wedge y_p x^{m-1} \wedge y_{i_1} x^m \wedge \cdots \wedge y_{i_q} x^m].$$

The corresponding ground state energy is $E(\boldsymbol{r}_0) = pm(m-1)+2qm$. When $k>1$, ground states wave functions are products of k determinants,

$$\psi_{\mathfrak{r}_0}(\boldsymbol{x},\boldsymbol{y}) = \Delta(\boldsymbol{x}) \prod_{\alpha=1}^{k} \mathcal{Y}_{\boldsymbol{r}_0^{(\alpha)}}(\boldsymbol{x},\boldsymbol{y}), \quad \mathfrak{r}_0 = (\boldsymbol{r}_0^{(1)},\cdots,\boldsymbol{r}_0^{(k)}).$$

The corresponding energy is simply $E(\mathfrak{r}_0) = kE(\boldsymbol{r}_0)$.

Other properties

- We show that the ground state wave functions obey the Knizhnik-Zamolodchikov equation,

$$(p+k)\partial_a \Phi_{\mathfrak{r}_0}(\boldsymbol{x},\boldsymbol{y}) = \sum_{b\neq a} \frac{J_{a,b}J_{b,a}}{x_a - x_b} \Phi_{\mathfrak{r}_0}(\boldsymbol{x},\boldsymbol{y}), \quad \Phi_{\mathfrak{r}_0}(\boldsymbol{x},\boldsymbol{y}) = \Delta(\boldsymbol{x})^{-1}\psi_{\mathfrak{r}_0}(\boldsymbol{x},\boldsymbol{y}).$$

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- Many known FQHE wave functions can be recovered from $\psi_{\mathfrak{r}}(\boldsymbol{x},\boldsymbol{y})$. For instance, taking $k=2$, $p=2$, $N=2m$, we find a singlet ground state $\mathfrak{r}_0 = (\boldsymbol{r}_0, \boldsymbol{r}_0)$ with $\boldsymbol{r}_0 = \{0, 1, \cdots, 2m-1\}$. The corresponding wave function reproduces the Moore-Read wave function.

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• Wedge states wave functions exhibits generalized statistics: $\mathcal{Y}_{\boldsymbol{r}_0}(\boldsymbol{x},\boldsymbol{y})$ do not vanish when two coordinates coincide $x_a \to x_b$ due to the presence of the spin variables $y_{i,a}$. But they do vanish when $p+1$ particles approach each other since at least two particles will have the same spin.

Free fermion formalism

- Realize wedge states for $k > 1$ as a (non-relativistic) free fermion correlator,

$$\left\{\psi_n^{i,\alpha}, \bar{\psi}_m^{j,\beta}\right\} = \delta_{i,j}\delta_{\alpha,\beta}\delta_{n,m}, \quad \left\{\psi_n^{i,\alpha}, \psi_m^{j,\beta}\right\} = \left\{\bar{\psi}_n^{i,\alpha}, \bar{\psi}_m^{j,\beta}\right\} = 0\,.$$

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- Use this description to see the emergence of Kac-Moody symmetry $\widehat{\mathfrak{su}(p)}_k$ in the large N limit.

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3. Quantum Hall Matrix Models

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$\Rightarrow$ In **[JEB-Matsuo 2401.03087]**, we diagonalized the matrix model and obtained a spin version of the Calogero Hamiltonian involving order k symmetric representations of $U(p)$.

DTT matrix model

- The matrix model introduced by **[Dorey, Tong, Turner 2016]** describes the moduli space of vortices. It involves a $N \times N$ complex matrix $Z(t)$ and p N-dimensional vectors $\varphi_i(t)$

$$\mathcal{S} = \int dt \left[\frac{1}{2} iB\, \mathrm{tr}(Z^\dagger \mathcal{D}_t Z) + i \sum_{i=1}^{p} \varphi_i^\dagger \mathcal{D}_t \varphi_i - (k+p)\, \mathrm{tr}\, \alpha - \omega\, \mathrm{tr}(Z^\dagger Z) \right]$$

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• The model also contains a non-dynamical gauge field α entering in the covariant derivatives $\mathcal{D}_t Z = \partial_t Z - i[\alpha, Z]$, $\mathcal{D}_t \varphi_i = \partial_t \varphi_i - i\alpha\varphi_i$. This gauge field imposes the constraint of the Gauss law

$$\frac{B}{2}[Z, Z^\dagger] + \sum_{i=1}^{p} \varphi_i \varphi_i^\dagger = (k+p)\mathbb{1}_N.$$

- In **[Dorey, Tong, Turner 2016]**, the complex matrix Z is diagonalized. Instead, we followed Polychronakos's original approach and decomposed the matrix Z in terms of Hermitian matrices,

$$Z = X_1 + iX_2, \quad Z^\dagger = X_1 - iX_2.$$

As a result, the action takes the form

$$\mathcal{S} = \int dt \left(-\frac{B}{2} \operatorname{tr} \left(X_1 \dot{X}_2 - X_2 \dot{X}_1 \right) + i \sum_{i=1}^{p} \varphi_i^\dagger \dot{\varphi}_i - \omega \operatorname{tr}(X_1^2 + X_2^2) + \operatorname{tr} \left[\alpha G(X_1, X_2, \varphi_i, \varphi_i^\dagger) \right] \right),$$

with $\quad G(X_1, X_2, \varphi_i, \varphi_i^\dagger) = -iB[X_1, X_2] + \sum_{i=1}^{p} \varphi_i \varphi_i^\dagger - (k+p)\mathbb{1}_N,$

Diagonalization

- The diagonalization is done following the usual method. Exploiting the $U(N)$ invariance, we decompose $X_1(t) = \Omega(t)^\dagger x(t)\Omega(t)$ with $(x)_{a,b} = x_a \delta_{a,b}$, $\Omega \in \mathrm{SU}(N)$. We derive the momenta and move to the Hamiltonian frame,

$$\mathcal{H} = \omega\,\mathrm{tr}(x^2 + (X_2)^2) = \omega \sum_{a=1}^{N} \left(x_a^2 + \frac{p_a^2}{B^2} \right) + 2\frac{\omega}{B^2} \sum_{a<b} \frac{(J_{a,b} - \Pi^+_{a,b})(J_{b,a} - \Pi^-_{a,b})}{(x_a - x_b)^2},$$

where $\Pi^\pm_{a,b}$ are momenta associated to angular degrees of freedom, and

$$J_{a,b} = \sum_{i=1}^{p} \varphi^\dagger_{i,a}\varphi_{i,b}, \quad p_a = \frac{\delta\mathcal{L}}{\delta\dot{x}_a} = BX_{2\ a,a},$$

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• Then, we impose the canonical quantization conditions. The gauge constraint imposed on physical states implies

$$\Pi^\pm_{a,b}\left|\mathrm{phys}\right\rangle = 0, \quad J_{a,a}\left|\mathrm{phys}\right\rangle = k\left|\mathrm{phys}\right\rangle.$$

$\rightsquigarrow$ **The parameter k identified to a mode number is quantized!**

- Due to the Vandermonde determinant in the measure coming from the diagonalization $dX_1 = d\Omega dx \Delta(\boldsymbol{x})$, the momentum p_a acts on wave functions as

$$p_a = -i\Delta(\boldsymbol{x})^{-1}\partial_a \Delta(\boldsymbol{x}), \quad \Delta(\boldsymbol{x}) = \prod_{\substack{a,b=1 \\ a<b}}^{N} (x_a - x_b).$$

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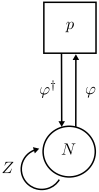
⚠ **In the following, set $B = 1$ upon rescaling of variables x_a.**

Quivers

- The matrix model describes a quantum mechanics associated to the quiver



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- Vertical arrows $\varphi \in \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^p)$,
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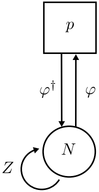

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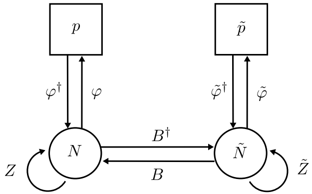
$\Rightarrow$ This quiver describes the moduli space of vortices of the Chern-Simons theory.

- A natural generalization of the previous quiver is the A_2 quiver



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$\rightsquigarrow$ We now have two real moment maps (Gauss laws)

$$\frac{1}{2}[Z, Z^\dagger] + \sum_{i=1}^{p} \varphi_i \varphi_i^\dagger + \sum_{c=1}^{\tilde{N}} B_{a,c} B^\dagger_{c,b} = (k + p + \tilde{N}) \mathbb{1}_N,$$

$$\frac{1}{2}[Z, Z^\dagger] + \sum_{i=1}^{p} \varphi_i \varphi_i^\dagger - \sum_{c=1}^{N} B^\dagger_{a,c} B_{c,b} = (\tilde{k} + \tilde{p}) \mathbb{1}_N.$$

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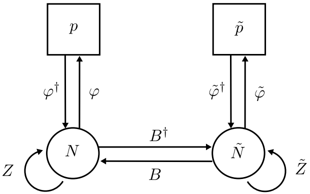

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$\Rightarrow$ Could this quiver describe states in the higher Landau levels?

A_2 Spin Calogero model

- After diagonalization, we find two Spin Calogero Hamiltonians coupled by the exchange of spin excitations,

$$\mathcal{H} = \left(\frac{1}{2} \sum_{a=1}^{N} \left(-\frac{\partial^2}{\partial x_a^2} + x_a^2 \right) + \sum_{a<b} \frac{(J_{a,b} - J^{B^\dagger}_{a,b})(J_{b,a} - J^{B^\dagger}_{b,a})}{(x_a - x_b)^2} \right)$$
$$+ \Lambda \left(\frac{1}{2} \sum_{a=1}^{\tilde{N}} \left(-\frac{\partial^2}{\partial y_a^2} + y_a^2 \right) + \sum_{a<b} \frac{(\tilde{J}_{a,b} + J^{B}_{a,b})(\tilde{J}_{b,a} + J^{B}_{b,a})}{(y_a - y_b)^2} \right),$$

where we denoted $\Lambda = \tilde{\omega}/\omega$, x_a and y_a are resp. the coordinates of vortices and excitations.

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where we denoted $\Lambda = \tilde{\omega}/\omega$, x_a and y_a are resp. the coordinates of vortices and excitations.

- The spin structure is given by the $\mathfrak{gl}(N)$ and $\mathfrak{gl}(\tilde{N})$ symmetry generators,

$$J_{a,b} = \sum_{i=1}^{p} \varphi^\dagger_{i,a} \varphi_{i,b}, \quad \tilde{J}_{a,b} = \sum_{i=1}^{\tilde{p}} \tilde{\varphi}^\dagger_{i,a} \tilde{\varphi}_{i,b},$$
$$J^B_{a,b} = \sum_{c=1}^{\tilde{N}} B^\dagger_{c,a} B_{b,c}, \quad J^{B\dagger}_{a,b} = \sum_{c=1}^{N} B^\dagger_{b,c} B_{c,a}.$$

- The gauge constraint suppress the angular degrees of freedom, and impose

$$\left(J_{a,a} - J^{B^\dagger}_{a,a}\right)|\text{phys}\rangle = k\,|\text{phys}\rangle\,, \quad \left(\tilde{J}_{a,a} + J^{B}_{a,a}\right)|\text{phys}\rangle = \tilde{k}\,|\text{phys}\rangle\,.$$

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$\Rightarrow$ Let's examine the abelian model more closely...

4. A_2 Calogero Model

Abelian model

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$\Rightarrow$ The Hamiltonian reduces to

$$\mathcal{H} = \left(\frac{1}{2} \sum_{a=1}^{N} \left(-\frac{\partial^2}{\partial x_a^2} + x_a^2 \right) + \sum_{a<b} \frac{(J_{a,b} - J_{a,b}^{B^\dagger})(J_{b,a} - J_{b,a}^{B^\dagger})}{(x_a - x_b)^2} \right)$$

$$+ \Lambda \left(\frac{1}{2} \sum_{a=1}^{\tilde{N}} \left(-\frac{\partial^2}{\partial y_a^2} + y_a^2 \right) + \sum_{a<b} \frac{J_{a,b}^B J_{b,a}^B}{(y_a - y_b)^2} \right),$$

with $J_{a,b} = \varphi_a^\dagger \varphi_b$, $J_{a,b}^B = \sum_{c=1}^{\tilde{N}} B_{c,a}^\dagger B_{b,c}$, $J_{a,b}^{B^\dagger} = \sum_{c=1}^{N} B_{b,c}^\dagger B_{c,a}$ ($\tilde{J} = 0$), and

$$\left(\varphi_a^\dagger \varphi_a - \sum_{b=1}^{\tilde{N}} B_{a,b}^\dagger B_{b,a} \right) |\mathrm{phys}\rangle = k\, |\mathrm{phys}\rangle\,, \quad \sum_{b=1}^{N} B_{b,a}^\dagger B_{a,b}\, |\mathrm{phys}\rangle = \tilde{k}\, |\mathrm{phys}\rangle\,.$$

Case of a single excitation

- We first restrict ourselves to a single excitation ($\tilde{N} = 1$),

$$J_{a,b} = \varphi_a^\dagger \varphi_b, \quad \tilde{J} = 0, \quad J_{a,b}^{B^\dagger} = B_b^\dagger B_a, \quad J^B = \sum_a B_a^\dagger B_a.$$

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- The Hamiltonian takes the form

$$\mathcal{H} = \mathcal{H}_0^{\mathrm{CM}}(\boldsymbol{x}) + \Lambda \mathcal{H}_0^{\mathrm{CM}}(y) + \sum_{a<b} \frac{(\varphi_a^\dagger \varphi_b - B_b^\dagger B_a)(\varphi_b^\dagger \varphi_a - B_a^\dagger B_b)}{(x_a - x_b)^2},$$

where we introduced the notation for the Calogero hamiltonian with coupling λ,

$$\mathcal{H}_\lambda^{\mathrm{CM}}(\boldsymbol{x}) = \frac{1}{2} \sum_a \left(-\frac{\partial^2}{\partial x_a^2} + x_a^2 \right) + \sum_{a<b} \frac{\lambda(\lambda+1)}{(x_a - x_b)^2}.$$

$\rightsquigarrow$ When $\lambda = 0$, we it reduces to decoupled harmonic oscillators.

- The constraints are

$$\sum_{a=1}^{N} B_a^\dagger B_a = \tilde{k}, \quad \varphi_a^\dagger \varphi_a - B_a^\dagger B_a = k,$$

⇝ Each particle carries a state $(B_a^\dagger)^{n_a}(\varphi_a^\dagger)^{n_a+k}\,|\emptyset\rangle$, with $n_a \geq 0$.

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- We deduce the action of the interaction term on the basis $\left|\boldsymbol{n}\right\rangle = \prod_a (B_a^\dagger)^{n_a}(\varphi_a^\dagger)^{n_a+k}\left|\emptyset\right\rangle$,

$$\begin{aligned}(\varphi_a^\dagger\varphi_b - B_b^\dagger B_a)(\varphi_b^\dagger\varphi_a - B_a^\dagger B_b)\left|\boldsymbol{n}\right\rangle = &((k+n_a)(k+n_b+1)+n_b(n_a+1))\left|\boldsymbol{n}\right\rangle \\ &- n_b(k+n_b)\left|\boldsymbol{n}+\delta_a-\delta_b\right\rangle - n_a(k+n_a)\left|\boldsymbol{n}-\delta_a+\delta_b\right\rangle\end{aligned}$$

$\rightsquigarrow$ This term can be diagonalized, leading to an expression of the Hamiltonian in different spin sectors.

$\Rightarrow$ Let's look at examples for small N!

- Two vortices and one excitation: a general spin state takes the form

$$|\boldsymbol{n}\rangle = |n_1, n_2\rangle = (B_1^\dagger)^{n_1}(\varphi_1^\dagger)^{n_1+k}(B_2^\dagger)^{n_2}(\varphi_2^\dagger)^{n_2+k}|0\rangle\,.$$

⇝ We find the following results for small $\tilde{k}$:

- $\tilde{k} = 0$: Only one spin state with $H_I\,|0,0\rangle = k(k+1)\,|0,0\rangle$, and so

$$\mathcal{H}(\boldsymbol{x},\boldsymbol{y})|_{|0,0\rangle} = \mathcal{H}_k^{\text{CM}}(\boldsymbol{x}) + \Lambda\mathcal{H}_0^{\text{CM}}(y).$$

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- $\tilde{k} = 1$: Two spin states $\{|1,0\rangle\,, |0,1\rangle\}$, the interaction term has two eigenvalues $k(k+1)$ and $(k+1)(k+2)$ with eigenstates $|\pm\rangle = |10\rangle \pm |01\rangle$,

$$\mathcal{H}(\boldsymbol{x},\boldsymbol{y})|_{|+\rangle} = \mathcal{H}_k^{\text{CM}}(\boldsymbol{x}) + \Lambda\mathcal{H}_0^{\text{CM}}(y), \quad \mathcal{H}(\boldsymbol{x},\boldsymbol{y})|_{|-\rangle} = \mathcal{H}_{k+1}^{\text{CM}}(\boldsymbol{x}) + \Lambda\mathcal{H}_0^{\text{CM}}(y).$$

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- $\tilde{k} = 2$: Interaction term has eigenvalues $k(k+1)$, $(k+1)(k+2)$, $(k+2)(k+3)$,

$$\mathcal{H}(\boldsymbol{x},\boldsymbol{y})|_{|1\rangle} = \mathcal{H}_k^{\mathrm{CM}}(\boldsymbol{x}) + \Lambda\mathcal{H}_0^{\mathrm{CM}}(y), \quad |1\rangle = |20\rangle + 2\,|11\rangle + |02\rangle$$

$$\mathcal{H}(\boldsymbol{x},\boldsymbol{y})|_{|0\rangle} = \mathcal{H}_{k+1}^{\mathrm{CM}}(\boldsymbol{x}) + \Lambda\mathcal{H}_0^{\mathrm{CM}}(y), \quad |0\rangle = |20\rangle - |02\rangle$$

$$\mathcal{H}(\boldsymbol{x},\boldsymbol{y})|_{|-1\rangle} = \mathcal{H}_{k+2}^{\mathrm{CM}}(\boldsymbol{x}) + \Lambda\mathcal{H}_0^{\mathrm{CM}}(y), \quad |-1\rangle = |20\rangle - 2\frac{k+2}{k+1}\,|11\rangle + |02\rangle\,.$$

• Three vortices and one excitation: We can no longer factorize the coordinate dependence in the interaction term,

$$\mathcal{H}(\boldsymbol{x},\boldsymbol{y}) = \mathcal{H}_0^{\mathrm{CM}}(\boldsymbol{x}) + \mathcal{H}_I(\boldsymbol{x}) + \Lambda\mathcal{H}_0^{\mathrm{CM}}(y), \quad \mathcal{H}_I = \sum_{a<b} \frac{H_{a,b}}{(x_a - x_b)^2}$$

$\rightsquigarrow$ We still have $\tilde{k} = \sum_a n_a$ acting as a natural grading:

- $\tilde{k} = 0$: Only one state and $H_{a,b}\,|0,0,0\rangle = k(k+1)\,|0,0,0\rangle$,

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- $\tilde{k} = 1$: Three states $\{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\}$, eigenvalues of the interaction term $\mathcal{H}_I$ are

$$k(k+1)\sum_{a<b}\frac{1}{x_{ab}^2}, \quad (k+1)^2\sum_{a<b}\frac{1}{x_{ab}^2} \pm \sqrt{R(\boldsymbol{x})},$$

$$\text{with} \quad R(\boldsymbol{x}) = \frac{3}{2}\sum_{a<b}\frac{1}{x_{ab}^4} - \frac{1}{2}\left(\sum_{a<b}\frac{1}{x_{ab}^2}\right)^2$$

$\rightsquigarrow$ For the first one, we recover again the Calogero-Moser system. For the others, it becomes very non-trivial!

Two vortices and two excitations

- In this case, we have four B matrices, with the commutators

$$[B_{11}, B_{11}^\dagger] = [B_{22}, B_{22}^\dagger] = [B_{12}, B_{21}^\dagger] = [B_{21}, B_{12}^\dagger] = 1.$$

Note that B_{12} is canonical conjugate to $B_{21}^\dagger$, and conversely B_{21} to $B_{12}^\dagger$.

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- From the constraints

$$\varphi_a^\dagger \varphi_a - \sum_{\tilde{c}=1,2} B_{a,\tilde{c}}^\dagger B_{\tilde{c},a} = k,$$

we deduce the general form of spin states

$$|\boldsymbol{n}\rangle = |\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}\rangle = (B_{11}^\dagger)^{n_{11}} (B_{12}^\dagger)^{n_{12}} (B_{21}^\dagger)^{n_{21}} (B_{22}^\dagger)^{n_{22}} (\varphi_1^\dagger)^{k+n_{11}+n_{12}} (\varphi_2^\dagger)^{k+n_{21}+n_{22}} |\emptyset\rangle \,.$$

$\rightsquigarrow$ Note that $\tilde{k} = n_{11} + n_{21} = n_{12} + n_{22}$ must be preserved.

- The hamiltonian can be decomposed into

$$\mathcal{H}(\boldsymbol{x},\boldsymbol{y}) = \mathcal{H}_0^{\rm CM}(\boldsymbol{x}) + \frac{\mathcal{H}_I}{x_{12}^2} + \Lambda\mathcal{H}_0^{\rm CM}(\boldsymbol{y}) + \Lambda\frac{\tilde{\mathcal{H}}_I}{y_{12}^2},$$

$$\text{with}\quad \mathcal{H}_I = (J_{12} - J_{12}^{B^\dagger})(J_{21} - J_{21}^{B^\dagger}),\quad \tilde{\mathcal{H}}_I = J_{12}^B J_{21}^B.$$

The expression of the action of interaction Hamiltonians on states $|\boldsymbol{n}\rangle$ reads

$$\begin{aligned}
\mathcal{H}_I\,|\boldsymbol{n}\rangle = {} & F(\boldsymbol{n})\,|\boldsymbol{n}\rangle - n_{21}(k+n_{21}+n_{22})\,|\boldsymbol{n}+\delta_{11}-\delta_{21}\rangle - n_{11}(k+n_{11}+n_{12})\,|\boldsymbol{n}-\delta_{11}+\delta_{21}\rangle \\
& - n_{22}(k+n_{21}+n_{22})\,|\boldsymbol{n}-\delta_{22}+\delta_{12}\rangle - n_{12}(k+n_{11}+n_{12})\,|\boldsymbol{n}-\delta_{12}+\delta_{22}\rangle \\
& + n_{11}n_{22}\,|\boldsymbol{n}-\delta_{11}-\delta_{22}+\delta_{12}+\delta_{21}\rangle + n_{12}n_{21}\,|\boldsymbol{n}+\delta_{11}+\delta_{22}-\delta_{12}-\delta_{21}\rangle \\
\tilde{\mathcal{H}}_I\,|\boldsymbol{n}\rangle = {} & \tilde{F}(\boldsymbol{n})\,|\boldsymbol{n}\rangle + n_{12}n_{21}\,|\boldsymbol{n}-\delta_{12}-\delta_{21}+\delta_{11}+\delta_{22}\rangle + n_{11}n_{22}\,|\boldsymbol{n}+\delta_{12}+\delta_{21}-\delta_{11}-\delta_{22}\rangle\,, \\
F(\boldsymbol{n}) = {} & (k+n_{11}+n_{12})(k+n_{21}+n_{22}+1) + n_{21}(n_{11}+1) + n_{22}(n_{12}+1), \\
\tilde{F}(\boldsymbol{n}) = {} & n_{11}(n_{12}+1) + n_{21}(n_{22}+1)
\end{aligned}$$

- $\tilde{k} = 0$: Single spin state, we find

$$\tilde{\mathcal{H}}_I \left| \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle = 0, \quad \mathcal{H}_I = \left| \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle = k(k+1) \left| \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle ,$$

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- $\tilde{k} = 1$: We have four spin states,

$$\left|\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right\rangle, \quad \left|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\rangle, \quad \left|\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\rangle, \quad \left|\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right\rangle.$$

$\rightsquigarrow$ We can find a common eigenbasis for $\mathcal{H}_I$ and $\tilde{\mathcal{H}}_I$,

$$\mathcal{H}(\boldsymbol{x},\boldsymbol{y})\big|_{|-\rangle} = \mathcal{H}_{k+1}^{\mathrm{CM}}(\boldsymbol{x}) + \Lambda \mathcal{H}_0^{\mathrm{CM}}(\boldsymbol{y}),$$

$$\mathcal{H}(\boldsymbol{x},\boldsymbol{y})\big|_{|a\rangle} = \mathcal{H}_{k}^{\mathrm{CM}}(\boldsymbol{x}) + \Lambda \mathcal{H}_1^{\mathrm{CM}}(\boldsymbol{y}),$$

$$\mathcal{H}(\boldsymbol{x},\boldsymbol{y})\big|_{|b\rangle} = \mathcal{H}_{k+1}^{\mathrm{CM}}(\boldsymbol{x}) + \Lambda \mathcal{H}_1^{\mathrm{CM}}(\boldsymbol{y}),$$

$$\mathcal{H}(\boldsymbol{x},\boldsymbol{y})\big|_{|c\rangle} = \mathcal{H}_{k+2}^{\mathrm{CM}}(\boldsymbol{x}) + \Lambda \mathcal{H}_1^{\mathrm{CM}}(\boldsymbol{y}).$$

5. Discussion

Summary on the A_2 model

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  - Is this model interesting?

    Need to perform a study for a general number of vortices N and at least two excitations!
  - Has this model any relevance for the Quantum Hall Effect?
    - $\rightsquigarrow$ What could be the interpretation of these excitations? Quasiholes?
    - $\rightsquigarrow$ Is there any relation to the second level of Jain's hierarchy?

Quasi-hole excitations

• The sum of two independent Calogero hamiltonian is reminiscent of the quasi-hole excitations. Starting from the corresponding wave function

$$\Psi(\boldsymbol{x},\boldsymbol{y}) = \Delta(\boldsymbol{x})^{\lambda}\Delta(\boldsymbol{y})^{1/\lambda}\prod_{a,b}(x_a - y_b)e^{-\frac{1}{2}(\sum_a x_a^2 + \lambda^{-1}\sum_a y_a^2)},$$

and rescaling $y_a \to \sqrt{\lambda} y_a$, i.e.

$$\Psi(\boldsymbol{x},\boldsymbol{y}) = \Delta(\boldsymbol{x})^{\lambda}\Delta(\boldsymbol{y})^{1/\lambda}\prod_{a,b}(x_a - \sqrt{\lambda} y_b)e^{-\frac{1}{2}(\sum_a x_a^2 + \sum_a y_a^2)},$$

we find that it is an eigenfunction of a sum of two Calogero Hamiltonian,

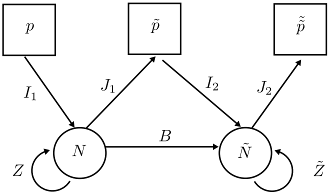
$$\mathcal{H}_{\mathrm{HL}}(\boldsymbol{x},\boldsymbol{y}) = \mathcal{H}_k^{\mathrm{CM}}(\boldsymbol{x}) + \mathcal{H}_{\tilde{k}}^{\mathrm{CM}}(\boldsymbol{y}), \quad k = \lambda - 1, \quad \tilde{k} = \lambda^{-1} - 1 = -\frac{k}{k+1}.$$

$\rightsquigarrow$ Reminiscent of what we obtained here in the simplest cases (provided $\Lambda = 1$).

Handsaw quiver

- Is this model merely a toy model for a more physical one?

E.g. the model one coming from handsaw quiver:



⇝ It has two real and one complex moment map:

$$\mu_{\mathbb{R}} = [Z, Z^\dagger] + B^\dagger B + I_1 I_1^\dagger - J_1^\dagger J_1,$$

$$\tilde{\mu}_{\mathbb{R}} = [\tilde{Z}, \tilde{Z}^\dagger] - BB^\dagger + I_2 I_2^\dagger - J_2^\dagger J_2,$$

$$\mu_{\mathbb{C}} = \tilde{Z}B - BZ + I_2 J_1.$$

- Having in mind the study of excitations upon vortices, we can restrict ourselves to the case $\tilde{p} = \tilde{\tilde{p}} = 0$. Then, the maps J_1, I_2 and J_2 become trivial, and only I_1 remains, which we denote φ. The two real moment map reduce to the ones we had before.

$\rightsquigarrow$ We recover our model, with an extra gauge constraint coming from the complex moment map:

$$G^{\mathbb{C}}_{a,b} = \sum_c \tilde{Z}_{a,c} B_{c,b} - \sum_c B_{a,c} Z_{c,b}.$$

$\rightsquigarrow$ How do we implement this new condition in the matrix model? In the Calogero model?

And more questions...

- Droplet model: Find corresponding Chern-Simons description of the matrix model? Explain possible physical applications.

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Thank you !!!

Quantum Algebras Meet

Gauge Theory and String Theory

January 12 - 16, 2026

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Chiung Hwang
Saebyeok Jeong
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Go Noshita
Andrei Okounkov
Jun'ichi Shiraishi
Eric Vasserot
Jun'ya Yagi
Masahito Yamazaki
Yegor Zenkevich
Keyou Zeng
(and more ...)

Organizers:

Arkadij Bojko, Jean-Emile Bourgine, Mykola Dedushenko, Zhengping Gui, Nafiz Ishtiaque, Tomoki Nosaka, Valerii Sopin, Yehao Zhou, Hao Zou

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