Fluctuation phenomena in entanglement dynamics

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L.-K. Lim, C.-Z. Lou, and **CT***, Nat. Commun. **15**, 1775 (2024) (+46 page Supplemental Materials)

1. **Motivations**
2. **Results**
   - analytical predictions
   - numerical verifications
3. **Mathematical theory**
   - ergodicity & emergent disorders
   - concentration of measure & modified logarithmic Sobolev inequality
   - statistics of temporal/out-of-equilibrium entanglement fluctuations

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2. **Results**
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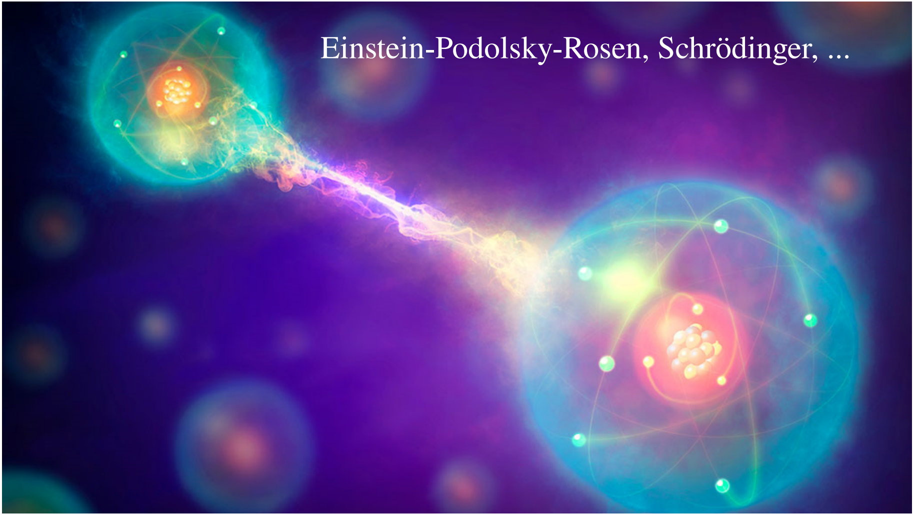
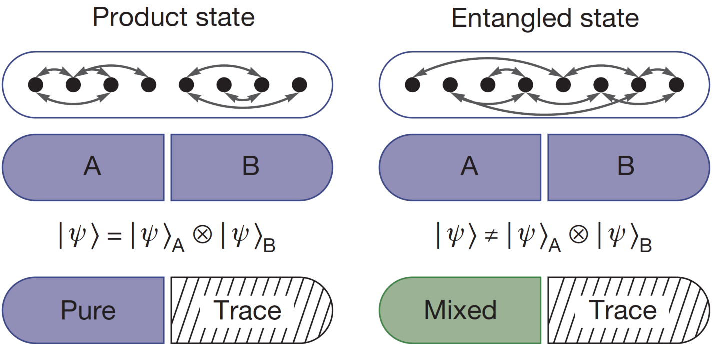
   - ergodicity & emergent disorders
   - concentration of measure & modified logarithmic Sobolev inequality
   - statistics of temporal/out-of-equilibrium entanglement fluctuations

Entanglement: a key concept in quantum physics and technology



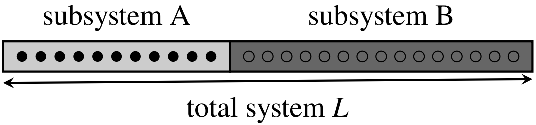
Entanglement: an elementary property of many-body state



Adapted from Greiner group's paper Nature **528**, 77–83 (2015)

- When no entanglement, subsystem state remains pure $\Rightarrow$ EE $= 0$
- When entangled, subsystem state is mixed $\Rightarrow$ EE> 0
- Maximally entangled $\Rightarrow$ EE$= \ln N$ (N: dim of subsystem Hilbert space)

Entanglement probes



- Pure state density matrix (total system):

$$\rho = |\Psi\rangle\langle\Psi|$$

- Reduced density matrix (subsystem A):

$$\rho_A = \mathrm{Tr}_B(\rho)$$

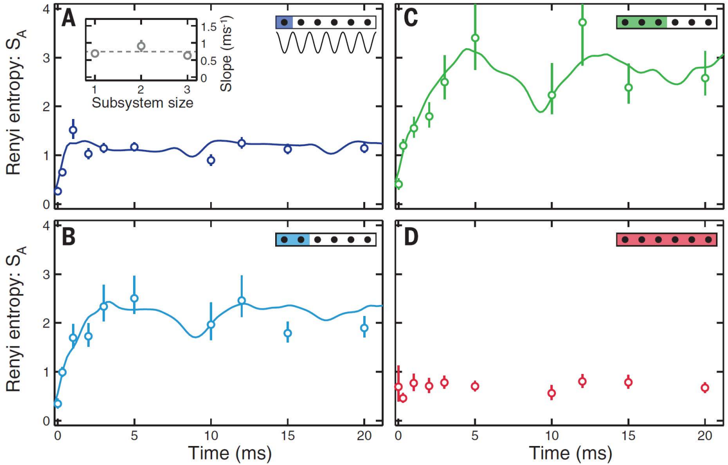
- Entanglement entropy (EE):

$$S = -\mathrm{Tr}_A(\rho_A \log \rho_A)$$

- nth order Rényi entropy

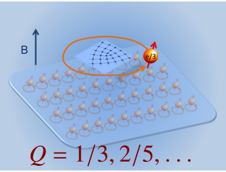
$$S_n = \frac{1}{1-n} \log \mathrm{Tr}_A(\rho_A^n)$$

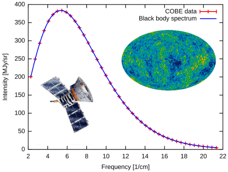
Ubiquitous entanglement noise



Greiner's group, Science **353**,794-800(2016)

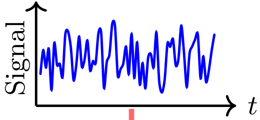
Why noise (temporal fluctuations) matter?

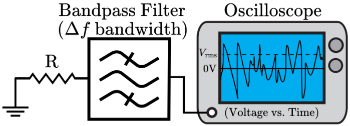




Fractional Quantum Hall Effect

Cosmic Microwave Background





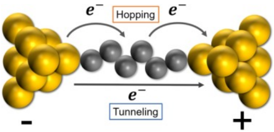
$$V_{\rm rms} = \sqrt{4k_B T R \Delta f}$$

Nyquist Noise

A key difference between entanglement and traditional observables

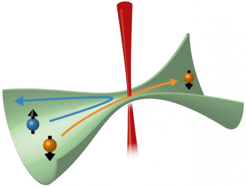
- symmetries vs. observables

charge transport



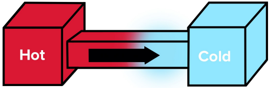
$U(1)$ symmetry

spin transport



$SU(2)$ symmetry

heat transport

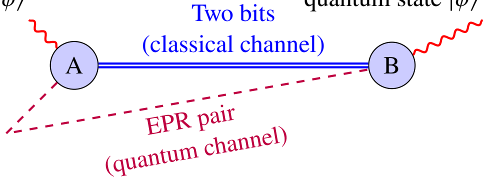


time translation symmetry

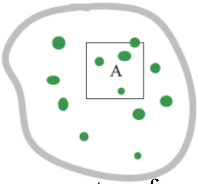
- information measures are not related to any symmetries

Photon with
quantum state $|\phi\rangle$

Photon with
quantum state $|\phi\rangle$



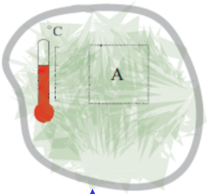
Pure-state statistical mechanics



quantum free gas
confined in a chaotic cavity

evolution of pure state $F(t)$

$$t_E = \lambda^{-1} \ln(A/\hbar)$$



Complex spatial structure of entanglement of indistinguishable particles

Thermalization!

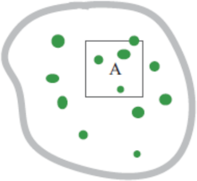
Traditional Statistical Physics:

- ~~ensembles~~
- ~~ergodicity~~
- ~~interactions~~

CANNOT thermalize

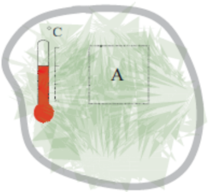
CT and K. Yang, Phys. Rev. B **104**, 174302 (2021)

Pure-state statistical mechanics



evolution of pure state $F(t)$

$$t_E = \lambda^{-1} \ln(A/\hbar)$$



$$\langle F(t) | a^\dagger_{\boldsymbol{r}'_1} \ldots a^\dagger_{\boldsymbol{r}'_i} a_{\boldsymbol{r}_1} \ldots a_{\boldsymbol{r}_1} | F(t) \rangle$$

$$\xrightarrow{t \gg t_E} \sum_P \sigma(P) \prod_{k=1}^{j} \frac{1}{V} \mathrm{d}m(\nu) J_0 \left(\frac{|\boldsymbol{r}_k - \boldsymbol{r}'_{P(k)}|}{\lambda_{\varepsilon_\nu}} \right) n_{\mathrm{FD}}(\varepsilon_\nu)$$

quantum expected value

||

thermal value

$$S_A(t) \xrightarrow{t \gg t_E} V_A S_0, \, S_0 = - \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi\hbar)^2} \left[n_{\mathrm{FD}} \left(\frac{\boldsymbol{p}^2}{2m} \right) \ln n_{\mathrm{FD}} \left(\frac{\boldsymbol{p}^2}{2m} \right) \left(1 - n_{\mathrm{FD}} \left(\frac{\boldsymbol{p}^2}{2m} \right) \right) \ln \left(1 - n_{\mathrm{FD}} \left(\frac{\boldsymbol{p}^2}{2m} \right) \right) \right]$$

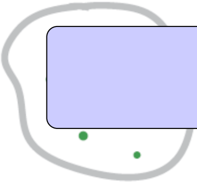
entanglement entropy

||

thermal entropy

CT and K. Yang, Phys. Rev. B **104**, 174302 (2021)

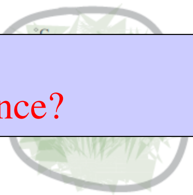
Pure-state statistical mechanics



Big Puzzle:

Where is quantum coherence?

$$t_E = \lambda^{-1}\ln(A/\hbar)$$



$$\langle F(t)|a^\dagger_{\boldsymbol{r}'_1}\ldots a^\dagger_{\boldsymbol{r}'_i}a_{\boldsymbol{r}_1}\ldots a_{\boldsymbol{r}_1}|F(t)\rangle$$

$$\xrightarrow{t\gg t_E}\sum_P \sigma(P)\prod_{k=1}^{j}\frac{1}{V}\mathrm{d}m(\nu)J_0\left(\frac{|\boldsymbol{r}_k-\boldsymbol{r}'_{P(k)}|}{\lambda_{\varepsilon_\nu}}\right)n_{\mathrm{FD}}(\varepsilon_\nu)$$

quantum expected value
||
thermal value

$$S_A(t)\xrightarrow{t\gg t_E}V_A S_0,\ S_0=-\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi\hbar)^2}\left[n_{\mathrm{FD}}\left(\frac{\boldsymbol{p}^2}{2m}\right)\ln n_{\mathrm{FD}}\left(\frac{\boldsymbol{p}^2}{2m}\right)\right.$$
$$\left.\left(1-n_{\mathrm{FD}}\left(\frac{\boldsymbol{p}^2}{2m}\right)\right)\ln\left(1-n_{\mathrm{FD}}\left(\frac{\boldsymbol{p}^2}{2m}\right)\right)\right]$$

entanglement entropy
||
thermal entropy

CT and K. Yang, Phys. Rev. B **104**, 174302 (2021)

Hint:
Temporal fluctuations of
macroscopic observable & observational entropy

Neumann, J.v. Beweis des Ergodensatzes und des *H*-Theorems in der neuen Mechanik. Z. Physik 57, 30–70 (1929)

für den Erwartungswert einer jeden makroskopisch meßbaren Größe A und für die Entropie im Zeitmittel:

$$\mathfrak{M}_t\left\{\left(E_A(U_\psi) - E_A(\psi_t)\right)^2\right\} \leqq \bar{\eta}^2 \cdot \mathrm{Max}_{a=1,2,\ldots}\left(\sum_{1}^{N_a}{}' \frac{S_a}{s_{\nu,a}}\left(\mathbf{M}_{\nu,a} + \mathbf{N}_{\nu,a}\right)\right),$$

$$\mathfrak{M}_t\left\{\left|S(U_\psi) - S(\psi_t)\right|\right\} \leqq \mathrm{Max}_{a=1,2,\ldots}\left(\sum_{1}^{N_a}{}' \frac{S_a}{s_{\nu,a}}\left(\mathbf{M}_{\nu,a} + \mathbf{N}_{\nu,a}\right)\right).$$

[Vgl. **II., 3.**; es genügt, das Max auf diejenigen a zu beziehen, deren

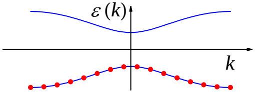
CT and K. Yang, Phys. Rev. B **104**, 174302 (2021)

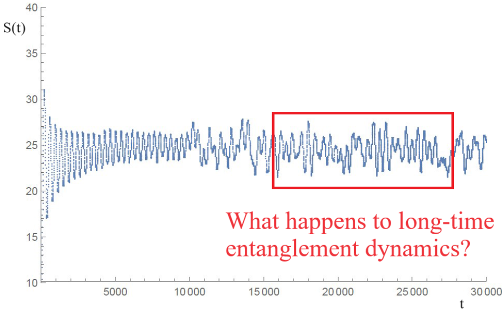
Formulation of problems: noninteracting

- Rice-Mele model

$$H_{\mathrm{RM}} = -\sum_{i=1}^{L}\left(J c_{i\bar{A}}^{\dagger} c_{i\bar{B}} + J' c_{i\bar{B}}^{\dagger} c_{(i+1)\bar{A}} + \mathrm{h.c.}\right) + M\sum_{i=1}^{L}\left(c_{i\bar{A}}^{\dagger} c_{i\bar{A}} - c_{i\bar{B}}^{\dagger} c_{i\bar{B}}\right)$$

- quench parameters (J, J', M): $(1, 0.5, 0.5) \to (1, 0.5, 1.5)$
- pre-quench state:



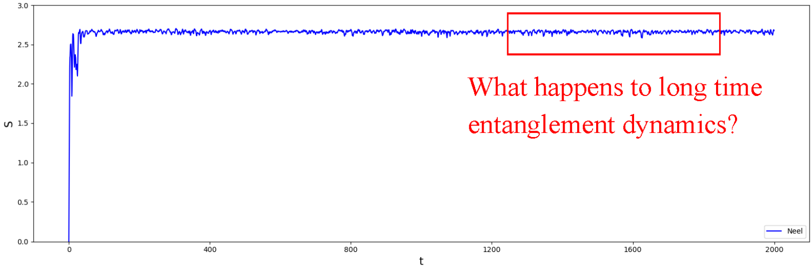


Formulation of problems: interacting

- spin-$\frac{1}{2}$ Heisenberg XXZ model

$$H_{XXZ} = J \sum_{i=1}^{L} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right)$$

- initial state: Neel state $|\uparrow\downarrow \ldots \uparrow\downarrow\rangle$



Fluctuation statistics of entanglement probes: noninteracting

- Models
  1. Rice-Mele model
  2. Transverse field Ising chain

$$H_{\mathrm{TFIC}} = -\frac{1}{2}\sum_{i=1}^{L}\left(\sigma_i^x\sigma_{i+1}^x + h\sigma_i^z\right)$$

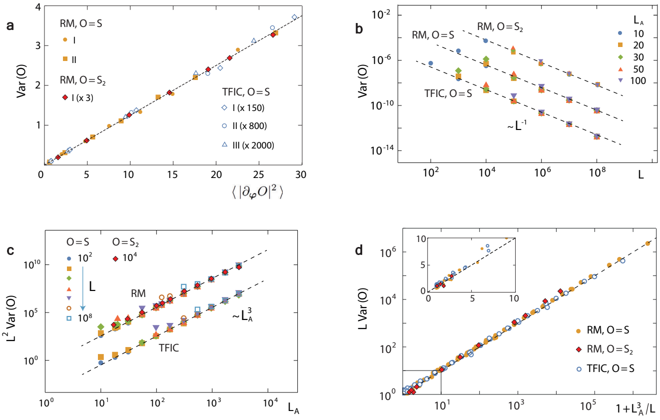
- Analytical predictions: universal w.r.t. system's details and probes
  1. Variance:
  $$\mathrm{Var}(O) = 1/L + L_A^3 \Big/ L^2\ ,\ O = S, S_2, \ldots$$
  2. Large deviation probability:

sub-Gaussian

sub-Gamma

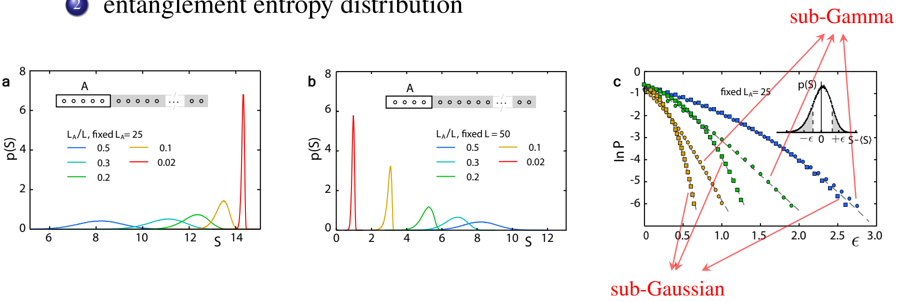
$$\mathbf{P}(|O - \langle O\rangle| \geq \epsilon) = \begin{cases} e^{-\frac{\epsilon^2}{2b_+}} & \text{for } O - \langle O\rangle > 0 \\ e^{-\frac{\epsilon^2}{2(b_- + c\epsilon)}} & \text{for } O - \langle O\rangle < 0 \end{cases}$$

- Numerical verifications
  1. Scaling law of the variance



Fluctuation statistics of entanglement probes: noninteracting

2. entanglement entropy distribution



Fluctuation statistics of entanglement probes: interacting

- model: spin-$\frac{1}{2}$ Heisenberg XXZ model
- Variance:

$$\mathrm{Var}(O) = L_A^{\beta} e^{-L}$$

- Large deviation probability:

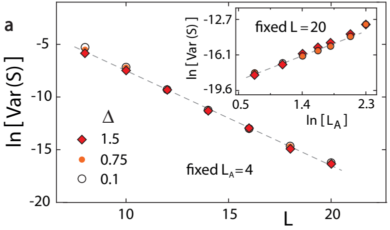
sub-Gaussian

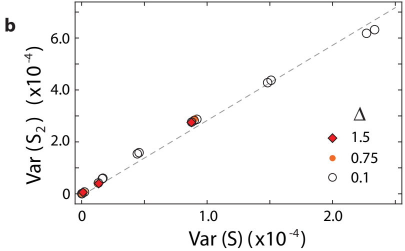
$$\mathbf{P}(|O - \langle O \rangle| \geq \epsilon) = \begin{cases} e^{-\frac{\epsilon^2}{2b_+}} & \text{for } O - \langle O \rangle > 0 \\ e^{-\frac{\epsilon^2}{2(b_- + c\epsilon)}} & \text{for } O - \langle O \rangle < 0 \end{cases}$$

sub-Gamma

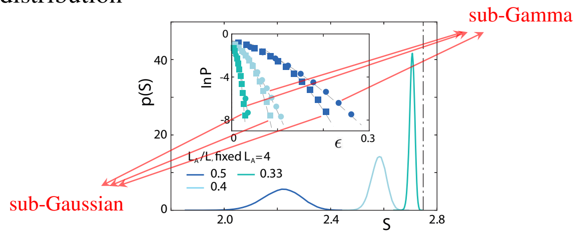
Fluctuation statistics of entanglement probes: interacting

- numerical verifications
  1. scaling law of variance

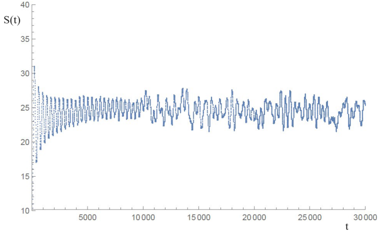
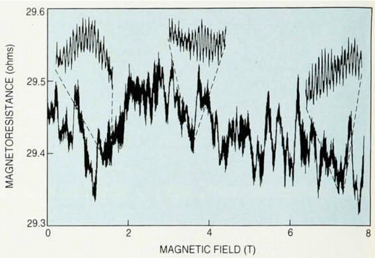




  2. full distribution



A new class of mesoscopic fluctuations

entanglement fluctuations	**mesoscopic conductance fluctuations**
	

disorder realization

$\boldsymbol{\varphi} = (\varphi_0, \varphi_1, \ldots, \varphi_{N-1})$	$\boldsymbol{V} \equiv (V_{\boldsymbol{r}_1}, V_{\boldsymbol{r}_2}, \ldots)$

distribution of disorder realizations

uniform in $\mathbb{T}^N$	Gaussian in $\mathbb{R}^N$

examples of probes

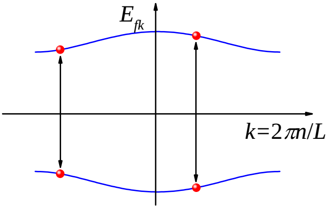
entanglement entropy, Rényi entropy	conductance, shot-noise power

- The matrix elements of the correlation matrix: $C_{ij} \equiv \langle \Psi(t)|c_i^\dagger c_j|\Psi(t)\rangle$

$$C_{j\sigma,j'\sigma'}(t) = \frac{1}{L}\sum_k e^{ik(j-j')}\left[\gamma_{\sigma\sigma'}(k) + \alpha_{\sigma\sigma'}(k)\cos\left(2E_{fk}t\right) + \beta_{\sigma\sigma'}(k)\sin\left(2E_{fk}t\right)\right]$$

Time-dependence enters here.

frequencies
associated with the energy gap
at each quasi-momentum k



Post-quench energy spectrum.

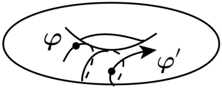
A key property of evolving observables

- The matrix elements of the correlation matrix: $C_{ij} \equiv \langle \Psi(t)|c_i^\dagger c_j|\Psi(t)\rangle$

$$C_{j\sigma,j'\sigma'}(t) = \frac{1}{L}\sum_k e^{ik(j-j')}\left[\gamma_{\sigma\sigma'}(k) + \alpha_{\sigma\sigma'}(k)\cos\left(2E_{fk}t\right) + \beta_{\sigma\sigma'}(k)\sin\left(2E_{fk}t\right)\right]$$

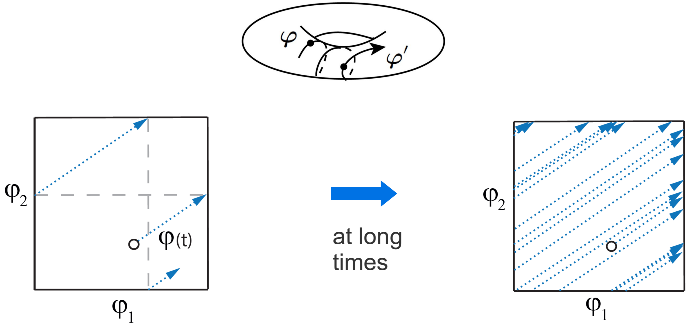
$$\boxed{C(t) = C_0 + C_1(\omega_1 t, \omega_2 t, \ldots, \omega_N t)}$$

- $N = L/2$ independent frequencies
- They enter as phase factors, so they are 2π periodic
- The quantum time-evolution is mapped onto a classical trajectory on the N-dimensional torus



Ergodicity and statistical equivalence

- Trajectory on N-torus:



- incommensurate $\omega_1, \omega_2, \ldots, \omega_N$,

$$\left\{ \begin{array}{c} x_1\omega_1 + x_2\omega_2 + \cdots + x_N\omega_N = 0 \\ \text{for integers } x_1, x_2, \ldots, x_N \end{array} \right. \Rightarrow x_1 = x_2 = \ldots = x_N = 0$$

Trajectory will be dense on the torus at long times.

Ergodicity and statistical equivalence

- **ergodic theorem**

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t\, f(\omega_1 t, \omega_2 t, \ldots, \omega_N t) = \int \frac{\mathrm{d}^N \varphi}{(2\pi)^N} f(\varphi_1, \varphi_2, \ldots, \varphi_N)$$

- incommensurate $\omega_1, \omega_2, \ldots, \omega_N$

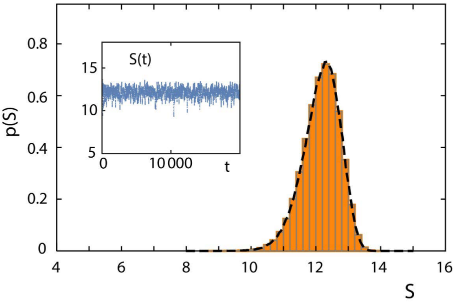
Statistics of the time series of the correlation matrix		Statistics of the ensemble of matrix
$C(t) = \widetilde{C}(\omega_1 t, \omega_2 t, \ldots, \omega_N t)$	$\Leftrightarrow$	$\widetilde{C}(\varphi_1, \varphi_2, \ldots, \varphi_N)$ with independent, uniformly drawn $\varphi_1, \varphi_2, \ldots, \varphi_N \in [0, 2\pi)$

Ergodicity and statistical equivalence: numerical verification

$$S(\varphi) = \int \mathrm{d}\lambda\, e(\lambda)\, \mathrm{Tr}_A\, \delta(\lambda - \widetilde{C}(\varphi))$$

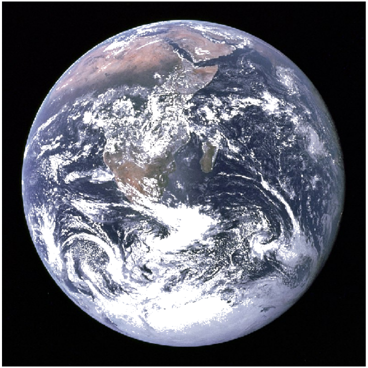
$$S(t) = \int \mathrm{d}\lambda\, e(\lambda)\, \mathrm{Tr}_A\, \delta(\lambda - C(t))$$

$$e(\lambda) = -\lambda \ln \lambda - (1-\lambda)\ln(1-\lambda)$$



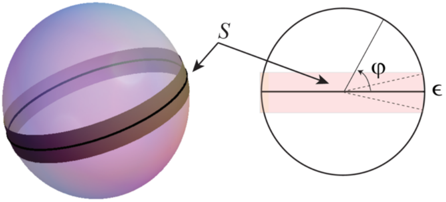
Histogram: $S(t)$. Dash line: $S(\varphi)$

The concentration of measure phenomenon: an example



- The area of the earth is $4\pi \times (6371\text{km})^2$.
- The area is uniformly distributed over the surface.
- What happens to the area of a high-dimensional "earth"?

The concentration of measure phenomenon: an example



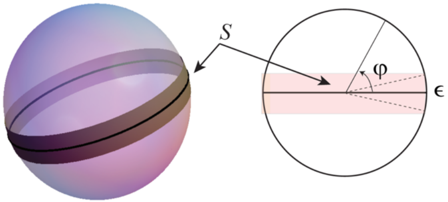
A unit sphere S^{2N-1} in $2N$ dimension and a strip around the equator of latitude $\epsilon/2$

The area measure of the strip:

$$\mu(S) = \frac{\int_{-\epsilon/2}^{\epsilon/2} \sin^{2N-2} \varphi \, \mathrm{d}\varphi}{\int_{-\pi/2}^{\pi/2} \sin^{2N-2} \varphi \, \mathrm{d}\varphi}$$

Let $\mu(S) = 0.9$, what's the width of the strip namely the value of ϵ?

The concentration of measure phenomenon: an example



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The area measure of the strip:

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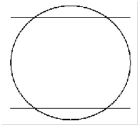
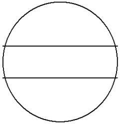
N	1	2	20	200	2000
ϵ/π	0.90	0.59	0.17	0.05	0.02

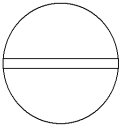
ϵ-values for $\mu(S) = 0.9$



$N = 2$

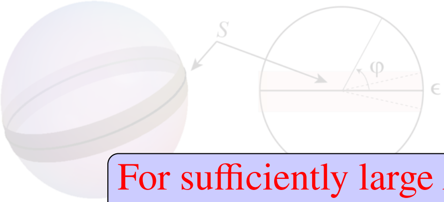


$N = 20$



$N = 200$

The concentration of measure phenomenon: an example



The area measure of the strip:

$$\frac{\int^{\epsilon/2} \sin^{2N-2}\varphi \, d\varphi}{\cdots d\varphi}$$

A unit spher

strip around

For sufficiently large N, the area of the high-dimensional sphere is concentrated around the equator!

N	1	2	20	200	2000
ϵ/π	0.90	0.59	0.17	0.05	0.02

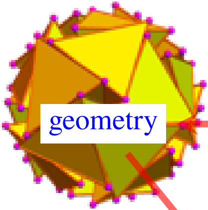
ϵ-values for $\mu(S) = 0.9$

$N = 2$

$N = 20$

$N = 200$

How to probe the concentration of measure?



a high-dimensional geometric body M

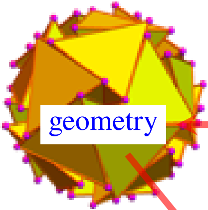
tool: real function $f : M \to \mathbb{R}$ with "nice" properties

$$\|f\|_{\text{Lip.}} = \sup_{c,c'} \frac{|f(c) - f(c')|}{d(c,c')} < \infty$$

"A random variable that depends (in a smooth way) on the influence of many independent variables (but not too much on any of them) is essentially constant."

– M. Talagrand

How to probe the concentration of measure?



geometry

a high-dimensional geometric body M

analysis

tool: real function $f : M \to \mathbb{R}$ with "nice" properties

$$\|f\|_{\mathrm{Lip.}} = \sup_{c,c'} \frac{|f(c) - f(c')|}{d(c,c')} < \infty$$

New perspectives of probability theory!

Modified logarithmic Sobolev inequality

- generic entanglement probe $O = S, S_2, \ldots$ as a highly nonlinear function over $\mathbb{T}^N$

$$O : \boldsymbol{\varphi} \equiv (\varphi_0, \varphi_1, \ldots, \varphi_{N-1}) \in \mathbb{T}^N \mapsto O(\boldsymbol{\varphi}) \in \mathbb{R}$$

- logarithmic moment-generating function

$$G(u) \equiv \ln \left\langle e^{u(O - \langle O \rangle)} \right\rangle, \; u \in \mathbb{R}$$

- modified logarithmic Sobolev inequality

$$\frac{\mathrm{d}}{\mathrm{d}u} \frac{G(u)}{u} \leq \frac{1}{u^2} \frac{\sum_{m=0}^{N-1} \left\langle \mathrm{e}^{u(O - \langle O \rangle)} \phi\left(-u\left(O - O_m\right)\right) \right\rangle}{\left\langle \mathrm{e}^{u(O - \langle O \rangle)} \right\rangle}$$

$$O_m : \boldsymbol{\varphi}'_m \equiv (\varphi_0, \varphi_1, \ldots, \varphi_{m-1}, \varphi_{m+1}, \ldots, \varphi_{N-1}) \in \mathbb{T}^{N-1} \mapsto O_m(\boldsymbol{\varphi}'_m) \in \mathbb{R}$$

The upper tail of the distribution

- a special class of O_m

$$O_m := \inf_{\varphi'_m} O\big(\varphi_0, \cdots, \varphi'_m, \cdots, \varphi_{N-1}\big)$$

- reduction of modified logarithmic Sobolev inequality

$$u > 0: \qquad \frac{\mathrm{d}}{\mathrm{d}u}\frac{G(u)}{u} \leq \frac{b_+}{2}, \; b_+ := \sum_{m=0}^{N-1} \left\langle (O - O_m^+)^2 \right\rangle$$

$$\Rightarrow \quad G(u) \leq \frac{b_+ u^2}{2}$$

- sub-Gaussian upper tail

$$\mathbf{P}(O - \langle O \rangle \geq \epsilon) \leq \mathrm{e}^{-\frac{\epsilon^2}{2b_+}}$$

The lower tail of the distribution

- a special class of O_m

$$O_m := \sup_{\varphi_m'} O\big(\varphi_0, \cdots, \varphi_m', \cdots, \varphi_{N-1}\big)$$

- reduction of modified logarithmic Sobolev inequality

$$u < 0 : \qquad \frac{\mathrm{d}}{\mathrm{d}u}\frac{G(u)}{u} \leq \frac{b_-}{2} + \delta F(u), \; b_- := \sum_{m=0}^{N-1} \big\langle (O - O_m^-)^2 \big\rangle$$

$$\delta F(u) \leq c_- \frac{\mathrm{d}G(u)}{\mathrm{d}u}, \; c_- < 0$$

$$\Rightarrow \quad G(u) \leq \frac{b_-}{2} \frac{u^2}{1 - c_- u}$$

- sub-Gamma lower tail

$$\mathbf{P}(\langle O \rangle - O \geq \epsilon) \leq \mathrm{e}^{-\frac{\epsilon^2}{2(b_- + |c_-|\epsilon)}}$$

Lipschitz continuity and fluctuations of entanglement probes

- Lipschitz continuity

$$|O(\boldsymbol{\varphi}) - O(\boldsymbol{\varphi}')| \leq \|O\|_{\text{Lip.}} \|\boldsymbol{\varphi} - \boldsymbol{\varphi}'\|, \ \forall \boldsymbol{\varphi}, \boldsymbol{\varphi}' \in \mathbb{T}^N$$

$$\text{Lipschitz constant: } \|O\|_{\text{Lip.}} = \sup_{\boldsymbol{\varphi} \in \mathbb{T}^N} \left\|\partial_{\boldsymbol{\varphi}} O\right\|$$

- relation between variance factor $b_\pm$ and a.e. Lipschitz continuity

$$b_\pm \propto \left\langle \left\|\partial_{\boldsymbol{\varphi}} O\right\| \right\rangle^2$$

- variance of entanglement probes

$$\text{Var}(O) \propto \left\langle \left\|\partial_{\boldsymbol{\varphi}} O\right\|^2 \right\rangle$$

Scaling law of the variance: noninteracting

- separation of the correlation matrix

nonrandom random

$$\tilde{C}(\varphi) = C_0 + C_1(\varphi)$$

matrix elements of C_1: $(C_1)_\ell \equiv \frac{1}{L} \sum_{m=1}^{N} [R_\ell(k_m) \cos\varphi_m + I_\ell(k_m) \sin\varphi_m]$

- separation of $\mathrm{Var}(S)$

$$\left.\begin{aligned} \partial_{\varphi_m} S &= \mathrm{Tr}\left[\mathcal{H}_A \partial_{\varphi_m} C_1\right] \\ \mathcal{H}_A &= \ln\left[\widetilde{C}^{-1}(\varphi) - \mathbb{I}\right] \end{aligned}\right\} \Rightarrow \mathrm{Var}(S) = \mathrm{Var}_1(S) + \mathrm{Var}_2(S)$$

edge contribution:

$$\mathrm{Var}_1(S) = \mathrm{const.} \sum_{m=1}^{N-1} \left\langle \left[\mathrm{Tr}_A \left(\ln\left(C_0^{-1} - \mathbb{I} \right) \partial_{\varphi_m} C_1 \right) \right]^2 \right\rangle$$

bulk contribution:

$$\mathrm{Var}_2(S) = \mathrm{const.} \sum_{m=1}^{N-1} \left\langle \left[\mathrm{Tr}_A \left(\left((\mathbb{I} - C_0) C_0 \right)^{-1} C_1 \partial_{\varphi_m} C_1 \right) \right] \right\rangle + \cdots$$

- separation of the correlation matrix

$$\tilde{C}(\varphi) = C_0 + C_1(\varphi)$$

matrix elements of C_1: $(C_1)_\ell \equiv \frac{1}{L}\sum_{m=1}^{N}\left[R_\ell(k_m)\cos\varphi_m + I_\ell(k_m)\sin\varphi_m\right]$

- separation of $\mathrm{Var}(S)$

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edge contribution:

$$\mathrm{Var}_1(S) = aL^{-1}$$

bulk contribution:

$$\mathrm{Var}_1(S) = b\, L_A^3 \Big/ L^2$$

Scaling law of the variance: interacting

- separation of RDM $\rho_A = \sum_{m,n} e^{-i(\omega_m-\omega_n)t}\chi_m\chi_n^* \mathrm{Tr}_B |\Psi_m\rangle\langle\Psi_n|$

nonrandom random

$$\widetilde{\rho}_A(\varphi) = \rho_{A0} + \widetilde{\rho}_{A1}(\varphi)$$

$$\widetilde{\rho}_{A1}(\varphi) = \sum_{m\neq n} e^{-i(\varphi_m-\varphi_n)}\chi_m\chi_n^* \mathrm{Tr}_B |\Psi_m\rangle\langle\Psi_n|$$

- nonrandom part of RDM ($1 \ll L_A \ll L$)

$$\rho_{A0} \simeq 1/D_A \ , \ (D_A : \text{dimension of subsystem Hilbert space})$$

- general expression of variance

$$\mathrm{Var}(O) \propto D_A^2 \left\langle \left| \mathrm{Tr}_A \left(\widetilde{\rho}_{A1}\partial_\varphi \widetilde{\rho}_{A1}\right)\right|^2 \right\rangle$$

$$\Downarrow$$

$$\text{scaling law: } \mathrm{Var}(O) \sim L_A^\beta e^{-\kappa L}, \ \kappa > 0, \ 2 \leq \beta \leq 4$$

- universal statistics of temporal/out-of-equilibrium fluctuations in entanglement dynamics
- a new class of mesoscopic fluctuations
- novel universality
  1. w.r.t. entanglement probes
  2. w.r.t. particle statistics (C.Z. Lou, **CT**, Z.X. Zou, T. Shi and L.K. Lim, to appear very soon)
- relations to black hole physics