Integrability with long-range interactions in extended Bose-Hubbard models.

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Extended Bose-Hubbard models

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$$H = \sum_{j=1}^{L} U_j N_j^2 + \sum_{j=1}^{L} \sum_{k \neq j}^{L} J_{jk} b_j^\dagger b_k.$$

is generally considered to not be integrable, even in the homogeneous case $U_j = U$, $J_{jk} = J$. It does conserve total particle number.

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$$H = \sum_{j=1}^{L} U_j N_j^2 + \sum_{j=1}^{L} \sum_{k \neq j}^{L} \frac{U_{jk}}{2} N_j N_k - \sum_{j=1}^{L} \sum_{k \neq j}^{L} J_{jk} b_j^\dagger b_k.$$

offer multiple integrable classes with additional conserved operators.

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offer multiple integrable classes with additional conserved operators.

Moreover, they are gaining prominence in experimental settings.

Extended Bose-Hubbard models

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https://doi.org/10.1088/1361-6633/adc3a7

Report on Progress

Recent progress on quantum simulations of non-standard Bose–Hubbard models

Titas Chanda[1,2,3], Luca Barbiero[4], Maciej Lewenstein[5,6], Manfred J Mark[7,8] and Jakub Zakrzewski[9,10,*]

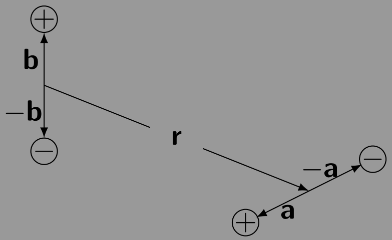
T. Chanda, L. Barbiero, M. Lewenstein, M.J. Mark, J. Zakrzewski, Rep. Prog. Phys. **88** (2025) 044501.

Dipolar interaction - electronic analogy

The potential between two electric charges q and q' takes the form

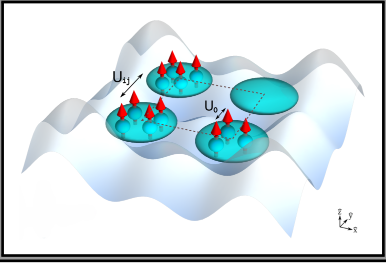
$$V(r) = \frac{Cqq'}{r}$$

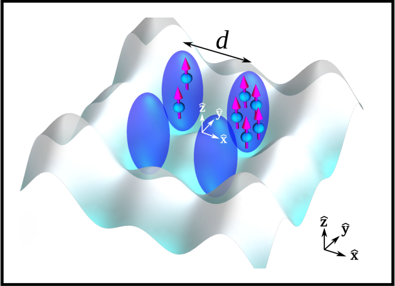
where r is the distance between the charges, and C is a constant.



An electric dipole is a separation of positive and negative charges. We can model a dipole by a rod of fixed length with a positive charge at one end, and a negative charge of the same magnitude on the other end.

$$V = Cq_a q_b \left(\frac{4\mathbf{a} \cdot \mathbf{b}}{r^3} - \frac{12(\mathbf{r} \cdot \mathbf{a})(\mathbf{r} \cdot \mathbf{b})}{r^5} \right).$$



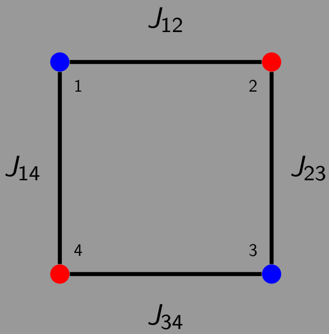


Top: oblate local potentials induce repulsive dipolar on-site interactions.
Bottom: prolate local potentials induce attractive dipolar on-site interactions.

Class 1 - obtained from the Yang-Baxter equation

Tonel, Ymai, Foerster, Links, J. Phys. A **48** (2015) 494001

$$\begin{aligned} H = & U(N_1+N_3-N_2-N_4)^2 + WN^2 \\ & + J_{12}(b_1^\dagger b_2 + b_2^\dagger b_1) + J_{23}(b_2^\dagger b_3 + b_3^\dagger b_2) \\ & + J_{34}(b_3^\dagger b_4 + b_4^\dagger b_3) + J_{14}(b_1^\dagger b_4 + b_4^\dagger b_1). \end{aligned}$$

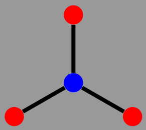


$$J_{12}J_{34} = J_{14}J_{23}$$

Class 2 - obtained from the Yang-Baxter equation, superintegrable

Ymai, Tonel, Foerster, Links, J. Phys. A **50** (2017) 264001
Bennett, Foerster, Isaac, Links, Nucl. Phys. B **998** (2024) 116406

$$H = U(N_1 + N_2 + N_3 - N_4)^2 \\ + J_1(b_1^\dagger b_4 + b_4^\dagger b_1) + J_2(b_2^\dagger b_4 + b_4^\dagger b_2) + J_3(b_3^\dagger b_4 + b_4^\dagger b_3).$$



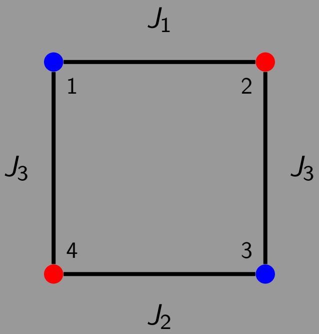
J_1, J_2, J_3 arbitrary

Three conserved operators close under commutation to realise the $sl(2)$ algebra.

Class 3 - obtained from the classical Yang-Baxter equation

Isaac, Links, Lukyanenko, Werry, arXiv:2507.00483

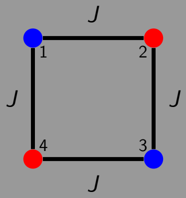
$$\begin{aligned} H = & U(N_1 + N_3 - N_2 - N_4)^2 \\ & + J_1(b_1^\dagger b_2 + b_2^\dagger b_1) + J_2(b_3^\dagger b_4 + b_4^\dagger b_3) \\ & + J_3(b_2^\dagger b_3 + b_3^\dagger b_2 + b_1^\dagger b_4 + b_4^\dagger b_1). \end{aligned}$$



Intersects with Class 1 when $J_1 J_2 = J_3^2$

Class 4 - superintegrable

$$H = U(N_1 - N_3)^2 + J(b_1^\dagger b_2 + b_2^\dagger b_1 + b_3^\dagger b_4 + b_4^\dagger b_3) \\ + J(b_1^\dagger b_4 + b_4^\dagger b_1 + b_2^\dagger b_3 + b_3^\dagger b_2).$$

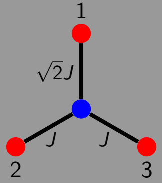


Conserved operators: N, $N_2 + N_4 - b_2^\dagger b_4 - b_4^\dagger b_2$,

$$e_0 = \frac{1}{4}(b_2^\dagger + b_4^\dagger)(b_2^\dagger + b_4^\dagger) - b_1^\dagger b_3^\dagger,$$
$$f_0 = \frac{1}{4}(b_2 + b_4)(b_2 + b_4) - b_1 b_3.$$

Class 5 - superintegrable

$$H = U(N_1 - N_2 - N_3)^2 + \sqrt{2}J(b_1^\dagger b_4 + b_4^\dagger b_1) + J(b_2^\dagger b_4 + b_4^\dagger b_2 + b_3^\dagger b_4 + b_4^\dagger b_3).$$



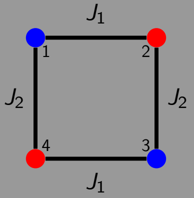
Conserved operators: N, $N_2 + N_3 - b_2^\dagger b_3 - b_3^\dagger b_2$,

$$e_0 = \frac{1}{2}(b_4^\dagger)^2 - \frac{1}{\sqrt{2}} b_1^\dagger (b_2^\dagger + b_3^\dagger),$$

$$f_0 = \frac{1}{2}(b_4)^2 - \frac{1}{\sqrt{2}} b_1 (b_2 + b_3).$$

Class 6 - superintegrable

$$H = H_1 + H_2 = U_1(N_1 - N_2 - N_3 + N_4)^2 + U_2(N_1 + N_2 - N_3 - N_4)^2$$
$$+ J_1(b_1^\dagger b_2 + b_2^\dagger b_1 + b_3^\dagger b_4 + b_4^\dagger b_3)$$
$$+ J_2(b_1^\dagger b_3 + b_3^\dagger b_1 + b_2^\dagger b_4 + b_4^\dagger b_2).$$



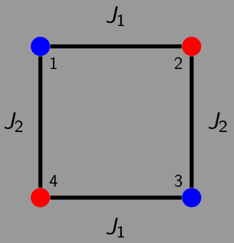
Conserved operators: N, H_1, H_2,

$$e_0 = b_1^\dagger b_3^\dagger - b_2^\dagger b_4^\dagger,$$
$$f_0 = b_1 b_3 - b_2 b_3.$$

Class 6 special case - only nearest-neighbour interactions

For $U_1 = U_2$, set

$$\begin{aligned}\bar{H} &= H + UN^2 \\ &= 2U(N_1^2 + N_2^2 + N_3^2 + N_4^2) + 2U(N_1 + N_3)(N_2 + N_4) \\ &\quad + J_1(b_1^\dagger b_2 + b_2^\dagger b_1 + b_3^\dagger b_4 + b_4^\dagger b_3) \\ &\quad + J_2(b_2^\dagger b_3 + b_3^\dagger b_2 + b_1^\dagger b_4 + b_4^\dagger b_1).\end{aligned}$$



Class 6 - Bethe Ansatz results

For N even let $n \in \{0, 2, \ldots, N\}$, and for N odd let $n \in \{1, 3, \ldots, N\}$. The Bethe Ansatz equations are

$$U \sum_{\substack{s=1 \\ s \neq k}}^{n} \frac{2u_k^2}{(u_k - u_s)} = U(n-1)u_k + J_1(u_k^2 - 1) \qquad k = 1, \ldots, n,$$

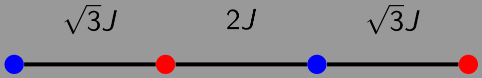
$$U \sum_{\substack{s=1 \\ s \neq k}}^{n} \frac{2v_k^2}{(v_k - v_s)} = U(n-1)v_k + J_2(v_k^2 - 1) \qquad k = 1, \ldots, n,$$

with energy eigenvalues $E = \frac{U}{2}\left(N^2 + n^2\right) - \sum_{k=1}^{n}(J_1 u_k + J_2 v_k)$.

The reference states are of the form $(e_0)^{(N-n)/2}|\mathrm{l.w.}(n)\rangle$ where $|\mathrm{l.w.}(n)\rangle$ is an $o(4)$ lowest-weight state.

Class 7 - open chain, superintegrable

$$
\begin{aligned}
H = U(3N_1 + N_2 - N_3 - 3N_4)^2 \\
&+ \sqrt{3}J(b_1^\dagger b_2 + b_2^\dagger b_1 + b_3^\dagger b_4 + b_4^\dagger b_3) \\
&+ 2J(b_2^\dagger b_3 + b_3^\dagger b_2).
\end{aligned}
$$



Conserved operators include

$$
\begin{aligned}
e_0 &= -3\sqrt{3}b_1^\dagger b_1^\dagger b_4^\dagger b_4^\dagger + \sqrt{3}b_2^\dagger b_2^\dagger b_3^\dagger b_3^\dagger + 6\sqrt{3}b_1^\dagger b_2^\dagger b_3^\dagger b_4^\dagger \\
&\quad - 4b_2^\dagger b_2^\dagger b_2^\dagger b_4^\dagger - 4b_1^\dagger b_3^\dagger b_3^\dagger b_3^\dagger \\
f_0 &= e_0^\dagger
\end{aligned}
$$

Cavity mediated interaction

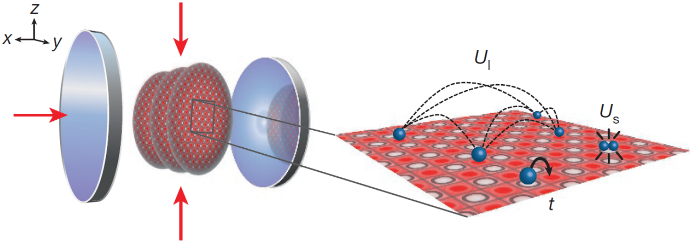


Figure 1 | Illustration of the experimental scheme that realizes a lattice model with on-site and infinite-range interactions. Left, a stack of 2D systems along the y axis is loaded into a 2D optical lattice (red arrows) between two mirrors (shown grey). The cavity induces atom–atom interactions of infinite range. Right, illustration of the competing energy scales: tunnelling t, on-site interactions U_s and long-range interactions U_l.

Landig, Hruby, Dogra et al., Nature **532** (2016) 476

wide range of the parameter space, the system can b
ttice model with long-range interactions (see Me
ed Data Fig. 1), given by:

$$\hat{H} = -t \sum_{\langle e,o \rangle} (\hat{b}_e^{\dagger} \hat{b}_o + \text{h.c.}) + \frac{U_{\text{s}}}{2} \sum_{i \in e,o} \hat{n}_i (\hat{n}_i - 1)$$

$$-\frac{U_{\text{l}}}{K} \left(\sum_e \hat{n}_e - \sum_o \hat{n}_o \right)^2 - \sum_{i \in e,o} \mu_i \hat{n}_i$$

and o denote all even and odd lattice sites respectivel

Landig, Hruby, Dogra et al., Nature **532** (2016) 476

Hamiltonian

Let $\{a_j,\, a_j^\dagger : j = 1, \ldots, m\} \cup \{b_j,\, b_j^\dagger : j = 1, \ldots, m\}$ denote mutually commuting sets of canonical boson operators satisfying

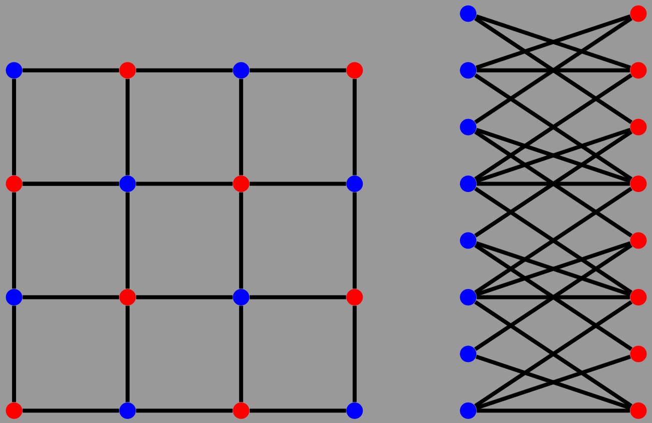
$$[a_j,\, a_k^\dagger] = [b_j,\, b_k^\dagger] = \delta_{jk} I,$$
$$[a_j,\, a_k] = [a_j^\dagger,\, a_k^\dagger] = [b_j,\, b_k] = [b_j^\dagger,\, b_k^\dagger] = 0.$$

For adjacency matrix $\mathcal{A} = \left(\begin{array}{ccc} 0 & | & \mathcal{B} \\ - & & - \\ \mathcal{B} & | & 0 \end{array} \right)$ the Hamiltonian reads

$$H = U(N_a - N_b)^2 + \sum_{j,k=1}^{m} \mathcal{B}_{jk}(a_j^\dagger b_k + b_j^\dagger a_k)$$

where $N_a = \sum_{j=1}^{m} a_j^\dagger a_j$, $N_b = \sum_{j=1}^{m} b_j^\dagger b_j$. The Hamiltonian admits a set of mutually-commuting conserved operators.

Example - open boundary conditions



Adjacency matrix

$$\mathcal{A} = \begin{pmatrix} 0 & | & \mathcal{B} \\ - & & - \\ \mathcal{B} & | & 0 \end{pmatrix} \cong \begin{pmatrix} \mathcal{B} & | & 0 \\ - & & - \\ 0 & | & -\mathcal{B} \end{pmatrix}.$$

Conserved operators

Explicitly, $[C(y),\, C(z)] = 0$ where

$$C(2p) = \sum_{j,k=1}^{m} \mathcal{B}_{jk}^{2p}(a_j^\dagger a_k + b_j^\dagger b_k),$$

$$C(2p+1) = U\sum_{i=0}^{2p} D(2p,i) + \sum_{j,k=1}^{m} \mathcal{B}_{jk}^{2p+1}(a_j^\dagger b_k + b_j^\dagger a_k)$$

with

$$D(2p,i) = \begin{cases} \displaystyle\sum_{j,k,r,q=1}^{m} \mathcal{B}_{jk}^{i}\mathcal{B}_{rq}^{2p-i}(a_j^\dagger a_q a_r^\dagger a_k + b_j^\dagger b_q b_r^\dagger b_k), & i \text{ even}, \\ \displaystyle\sum_{j,k,r,q=1}^{m} (\mathcal{B}_{jk}^{i}\mathcal{B}_{rq}^{2p-i} + \mathcal{B}_{jk}^{2p-i}\mathcal{B}_{rq}^{i})a_j^\dagger a_q b_r^\dagger b_k, & i \text{ odd}. \end{cases}$$

Canonical transformation

Let X denote a unitary operator that diagonalises $\mathcal{B}$, viz.

$$\sum_{p=1}^{m} X^\dagger_{jp} X_{pk} = \delta_{jk}, \qquad \sum_{p,q=1}^{m} X^\dagger_{jp} \mathcal{B}_{pq} X_{qk} = \varepsilon_j \delta_{jk},$$

with $\{\varepsilon_j : j = 1, ..., m\}$ the spectrum of $\mathcal{B}$. Introducing

$$a_k = \sum_{j=1}^{m} X_{kj} c_j, \quad b_k = \sum_{j=1}^{m} X_{kj} d_j, \quad a^\dagger_k = \sum_{j=1}^{m} X^\dagger_{jk} c^\dagger_j, \quad b^\dagger_k = \sum_{j=1}^{m} X^\dagger_{jk} d^\dagger_j,$$

leads to $N_a = \sum_{j=1}^{m} c^\dagger_j c_j = N_c$, $N_b = \sum_{j=1}^{m} d^\dagger_j d_j = N_d$ and

$$H = U(N_c - N_d)^2 + \sum_{j=1}^{m} \varepsilon_j (c^\dagger_j d_j + d^\dagger_j c_j).$$

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$$a_k = \sum_{j=1}^{m} X_{kj} c_j, \quad b_k = \sum_{j=1}^{m} X_{kj} d_j, \quad a_k^{\dagger} = \sum_{j=1}^{m} X_{jk}^{\dagger} c_j^{\dagger}, \quad b_k^{\dagger} = \sum_{j=1}^{m} X_{jk}^{\dagger} d_j^{\dagger},$$

leads to $N_a = \sum_{j=1}^{m} c_j^{\dagger} c_j = N_c$, $N_b = \sum_{j=1}^{m} d_j^{\dagger} d_j = N_d$ and
$H = U(N_c - N_d)^2 + \sum_{j=1}^{m} \varepsilon_j (c_j^{\dagger} d_j + d_j^{\dagger} c_j)$. Note that
$\hat{N}_j = c_j^{\dagger} c_j + d_j^{\dagger} d_j$ are conserved operators; let N_j denote their eigenvalues. Then $\sum_{j=1}^{m} N_j = N$ is the total number of particles.

Bethe Ansatz results

For $\{N_1, \ldots, N_m : N_j \in \mathbb{Z}_{\geq 0}\}$ set $|N_1, \ldots, N_m\rangle = (d_1^\dagger)^{N_1} \ldots (d_m^\dagger)^{N_m}|0\rangle$ where $|0\rangle$ denotes the vacuum. The energy eigenvalues are

$E = UN^2 + 4U \sum_{j=1}^{m} \sum_{n=1}^{N} \frac{N_j \varepsilon_j^2}{v_n - \varepsilon_j^2}$ subject to

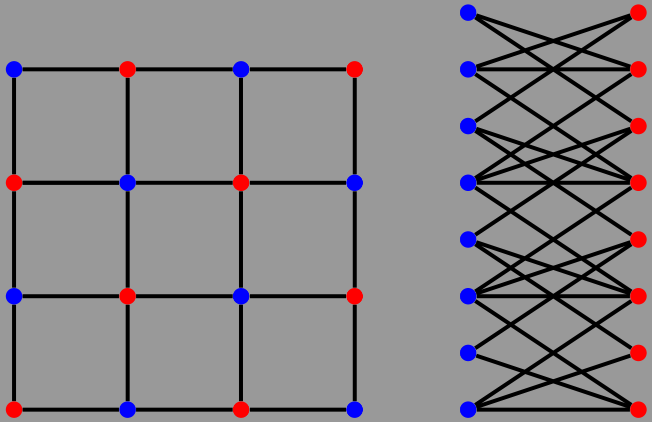
$$\sum_{m \neq n}^{N} \frac{2v_n}{v_n - v_m} + \frac{\prod_{j=1}^{m}(v_n - \varepsilon_j^2)^{N_j}}{16U^2 \prod_{m \neq n}^{N}(v_n - v_m)} = N - 1 + \sum_{j=1}^{m} \frac{N_j \varepsilon_j^2}{v_n - \varepsilon_j^2}.$$

for $n = 1, \ldots, N$. The Bethe eigenstates read

$$|v_1, \ldots, v_N; N_1, \ldots, N_m\rangle = \prod_{n=1}^{N} C(v_n)|N_1, \ldots, N_m\rangle,$$

$$C(u) = \frac{1}{2U} I + \sum_{j=1}^{m} \frac{2\varepsilon_j}{u - \varepsilon_j^2} c_j^\dagger d_j.$$

Example - open boundary conditions



Adjacency matrix

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Adjacency matrix spectrum

The eigenvalues of the adjacency matrix for open boundary conditions are of the form

$$2\cos\left(\frac{\pi j}{L+1}\right)+2\cos\left(\frac{\pi k}{L+1}\right) \qquad j,k\in\{1,\ldots,L\}.$$

The eigenvalues of the adjacency matrix for cylindrical boundary conditions are of the form

$$2\cos\left(\frac{2\pi j}{L}\right)+2\cos\left(\frac{\pi k}{L+1}\right) \qquad j,k\in\{1,\ldots,L\}.$$

The eigenvalues of the adjacency matrix for toroidal boundary conditions are of the form

$$2\cos\left(\frac{2\pi j}{L}\right)+2\cos\left(\frac{2\pi k}{L}\right) \qquad j,k\in\{1,\ldots,L\}.$$

These provide the single quasi-particle energies.

Degeneracies lead to superintegrability

Without loss of generality, the Hamiltonian

$$H = U(N_c - N_d)^2 + \sum_{j=1}^{m} \varepsilon_j (c_j^\dagger d_j + d_j^\dagger c_j)$$

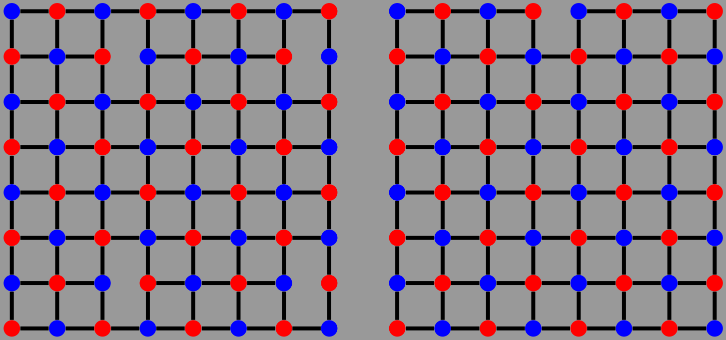
can be brought into a form where all $\varepsilon_j \geq 0$ through transformations of the type $d_j \mapsto -d_j$.

If $\varepsilon_j = \varepsilon_{j+1} = \dots \varepsilon_{j+r-1}$ then for $s, t \in \{0, \dots r-1\}$ the operators

$$\begin{gathered} c_{j+s}^\dagger c_{j+t} + d_{j+s}^\dagger d_{j+t}, \\ c_{j+s}^\dagger d_{j+t}^\dagger - d_{j+s}^\dagger c_{j+t}^\dagger, \\ c_{j+s} d_{j+t} - d_{j+s} c_{j+t} \end{gathered}$$

commute with H, and close to form the Lie algebra $o(2r)$.

Models with defects



(a) (b)

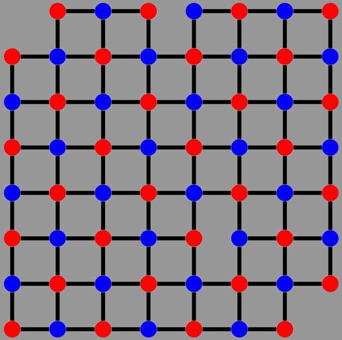
These models with defects are integrable, due to the preservation of reflection symmetry.

In the case of (b) on the previous slide, the characteristic polynomial $P(\lambda)$ for $\mathcal{B}$ reads

$$
\begin{aligned}
P(\lambda) = \lambda^{32} - 7\lambda^{31} - 31\lambda^{30} + 309\lambda^{29} + 217\lambda^{28} - 5679\lambda^{27} \\
+ 3187\lambda^{26} + 56798\lambda^{25} - 70399\lambda^{24} - 338714\lambda^{23} \\
+ 587333\lambda^{22} + 1234858\lambda^{21} - 2759107\lambda^{20} - 2672048\lambda^{19} \\
+ 7975967\lambda^{18} + 2956747\lambda^{17} - 14533255\lambda^{16} - 299377\lambda^{15} \\
+ 16620705\lambda^{14} - 3246561\lambda^{13} - 11634583\lambda^{12} + 3872804\lambda^{11} \\
+ 4737597\lambda^{10} - 2021218\lambda^{9} - 1010163\lambda^{8} + 512874\lambda^{7} \\
+ 85077\lambda^{6} - 55422\lambda^{5} + 324\lambda^{4} + 1296\lambda^{3},
\end{aligned} \tag{1}
$$

and the characteristic polynomial of $\mathcal{A}$ is $P(\lambda)P(-\lambda)$. Use of the computer algebra package MAGMA, hosted by The University of Sydney, indicates that (1) is not solvable in the Galois sense. That is, the roots cannot be expressed in terms of radicals. The only resort is numerical computation. However, the associated Hamiltonian is superintegrable.

Models with defects



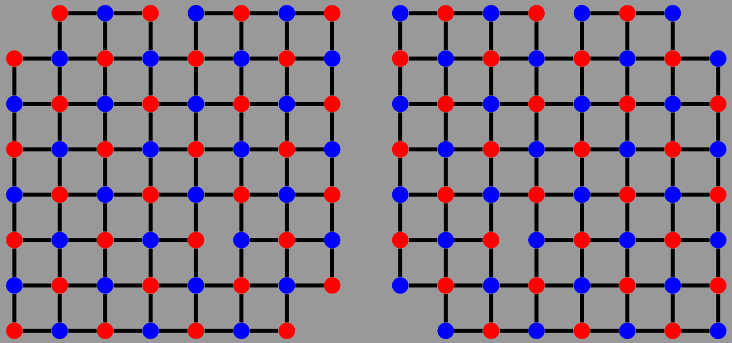
Reflection symmetry is broken. In particular the number of blue vertices does not equal the number of red vertices.

The matrix $\mathcal{B}$ is no longer square. We remedy the situation by making a reflection copy and consider an enhanced system. The adjacency matrix reads

$$\mathcal{A} = \begin{pmatrix} 0 & \mathcal{B} & | & 0 & 0 \\ \mathcal{B}^T & 0 & | & 0 & 0 \\ - & - & & - & - \\ 0 & 0 & | & 0 & \mathcal{B} \\ 0 & 0 & | & \mathcal{B}^T & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 0 & | & 0 & \mathcal{B} \\ 0 & 0 & | & \mathcal{B}^T & 0 \\ - & - & & - & - \\ 0 & \mathcal{B} & | & 0 & 0 \\ \mathcal{B}^T & 0 & | & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & | & \mathcal{C} \\ - & & - \\ \mathcal{C} & | & 0 \end{pmatrix}$$

where $\mathcal{C}$ is symmetric.

Models with defects



While the individual latices do not have reflection symmetry, the total system does. Setting the particle number to zero for one subsystem provides conserved operators for the other. Obtaining a Bethe Ansatz solution for this scenario uses a Singular Value Decomposition for $\mathcal{B}$.

Generalised conserved operators

$$C(2p) = \sum_{j,k=1}^{r} (\mathcal{B}\mathcal{B}^T)^p_{jk} a^\dagger_j a_k + \sum_{j,k=1}^{s} (\mathcal{B}^T\mathcal{B})^p_{jk} b^\dagger_j b_k,$$

$$C(2p+1) = U\sum_{i=0}^{2p} D(2p,i) + \sum_{j=1}^{r}\sum_{k=1}^{s} (\mathcal{B}(\mathcal{B}^T\mathcal{B})^p)_{jk} a^\dagger_j b_k + \sum_{j=1}^{r}\sum_{k=1}^{s} (\mathcal{B}^T(\mathcal{B}\mathcal{B}^T)^p)_{kj} b^\dagger_k a_j$$

Generalised conserved operators - continued

with

$$D(2p,i)=\begin{cases}\displaystyle\sum_{j,k,l,q=1}^{r}(\mathcal{B}\mathcal{B}^T)^{i/2}_{jk}(\mathcal{B}\mathcal{B}^T)^{p-i/2}_{lq}a_j^\dagger a_q a_l^\dagger a_k \\ \displaystyle\quad+\sum_{j,k,l,q=1}^{s}(\mathcal{B}^T\mathcal{B})^{i/2}_{jk}(\mathcal{B}^T\mathcal{B})^{p-i/2}_{lq}b_j^\dagger b_q b_l^\dagger b_k, & i\ \text{even},\\ \displaystyle\sum_{j,q=1}^{r}\sum_{l,k=1}^{s}(\mathcal{B}(\mathcal{B}^T\mathcal{B})^{(i-1)/2})_{jk}(\mathcal{B}^T(\mathcal{B}\mathcal{B}^T)^{p-1-(i-1)/2})_{lq} \\ \quad\quad\times a_j^\dagger a_q b_l^\dagger b_k \\ \displaystyle\quad+\sum_{j,q=1}^{r}\sum_{l,k=1}^{s}(\mathcal{B}(\mathcal{B}^T\mathcal{B})^{p-1-(i-1)/2})_{jk}(\mathcal{B}^T(\mathcal{B}\mathcal{B}^T)^{(i-1)/2})_{lq} \\ \quad\quad\times a_j^\dagger a_q b_l^\dagger b_k, & i\ \text{odd}.\end{cases}$$

Summary

- Extended Bose-Hubbard models offer integrability for special choices of coupling parameters.
- Small systems with few degrees of freedom provide candidates for modelling systems with long-range magnetic dipole interactions.
- For global range cavity-mediated interactions on the square lattice, exact solutions are available for open, cylindrical, and toroidal bounday condistions.
- The integrability of these global-range interactions models is robust with respect to defects. The impacts with respect to Bethe Ansatz solutions are more nuanced. This is work in progress, based on implementing singular value decompositions of the adjacency submatrix $\mathcal{B}$, and understanding the required pseudo-vacuum states.
- Superintegrability was identified across a range of examples for both interaction types.

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