

Recent Advances in Rational Q -system

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@Challenges in Integrability, Wuhan

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Based on the works

J. Hou, **Y. Jiang** and Y. Miao, *SciPost Phys.* 16 (2024) 5, 129

Y. Jiang, Y. Liu, Y. Miao and Z. Tan, *to appear*

[Also see the poster by Liu (柳艺超) and Tan (谭子曦)]

Bethe ansatz

Heisenberg spin chain

$$H_{\text{XX}} = J \sum_{i=1}^L \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z \right)$$

Bethe ansatz

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Bethe ansatz equation

Eigenstate: $|\{u_j\}_{j=1}^N\rangle$

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} \right)^L = \prod_{\substack{k=1 \\ k \neq j}}^N \frac{u_j - u_k + i}{u_j - u_k - i}, \quad j = 1, \dots, N$$

Bethe ansatz

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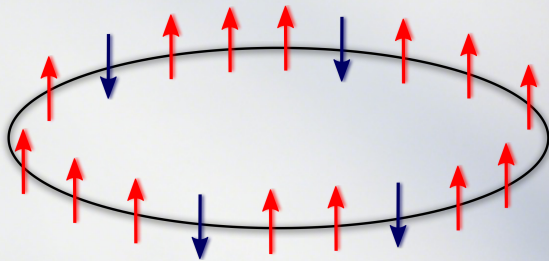
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Task: Solve BAE and find Bethe roots

Physical properties

- Ground state
- Excitations
- Thermodynamics: entropy, free energy, ...
- Kondo effect

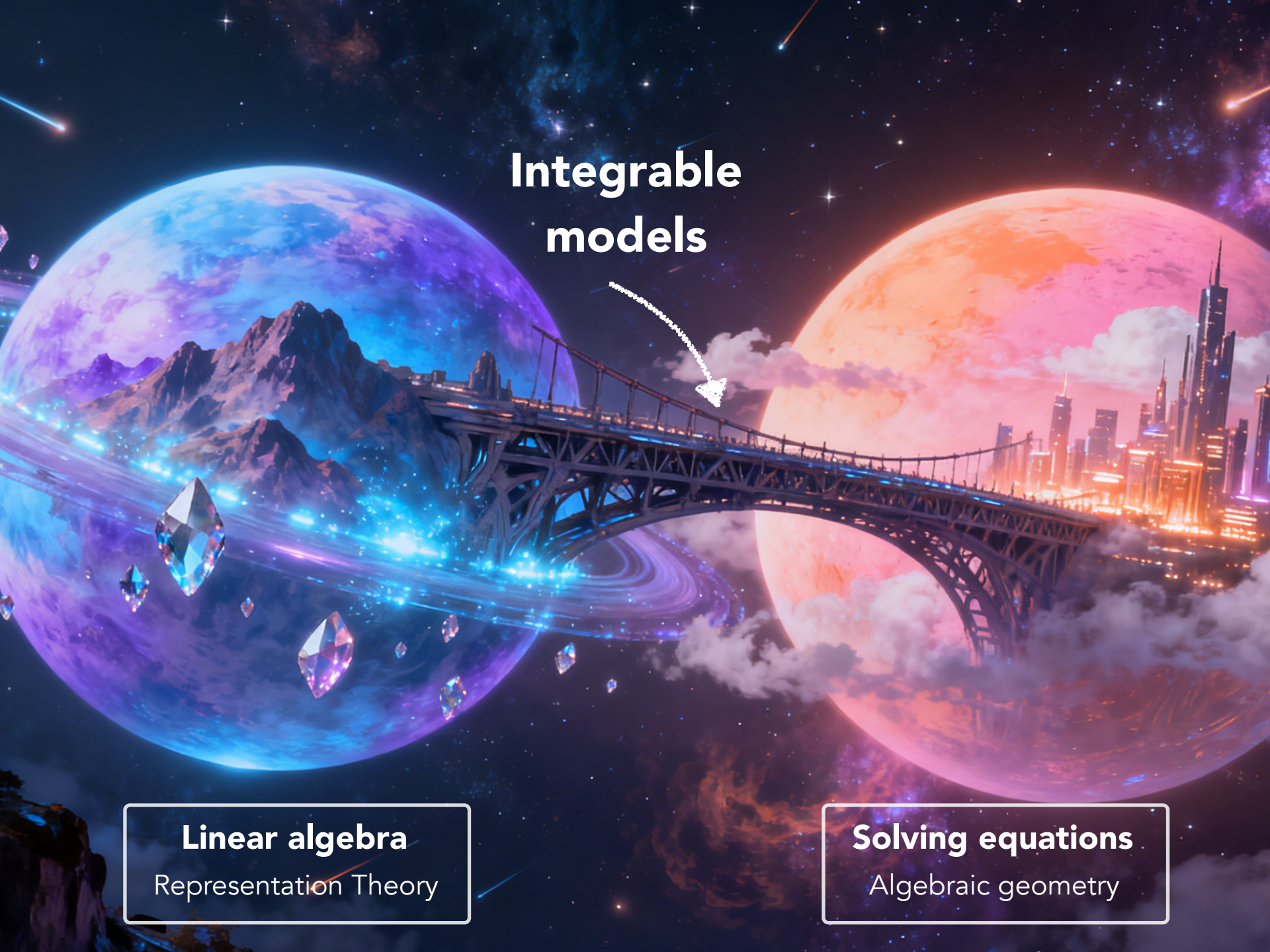


Solutions of BAE

- A specific solution of BAE
- Particle-hole pair, Bethe strings
- Density distribution of the Bethe roots
- Complex Bethe root



Integrable models

A surreal space landscape. A large bridge with a suspension structure spans across the frame. On the left, a large blue planet with a mountain range is visible. On the right, a large orange planet with a city skyline is visible. The bridge connects the two planets. Several large, faceted diamonds are floating in the space between the planets. A white dotted arrow points from the text 'Integrable models' to the bridge.

Linear algebra
Representation Theory

Solving equations
Algebraic geometry

Specific solutions

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Example 1. Anti-ferromagnetic vacuum

In condensed matter, one typically cares about **anti-ferromagnetic ground** state, and **low energy excitations**

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Numerical method

Logarithmic of the BAE

$$2L \arctan(u_j) = 2\pi I_j + \sum_{k \neq j}^N \arctan(u_j - u_k), \quad j = 1, \dots, N$$

Specific solutions

Example 2. Semi-classical solution

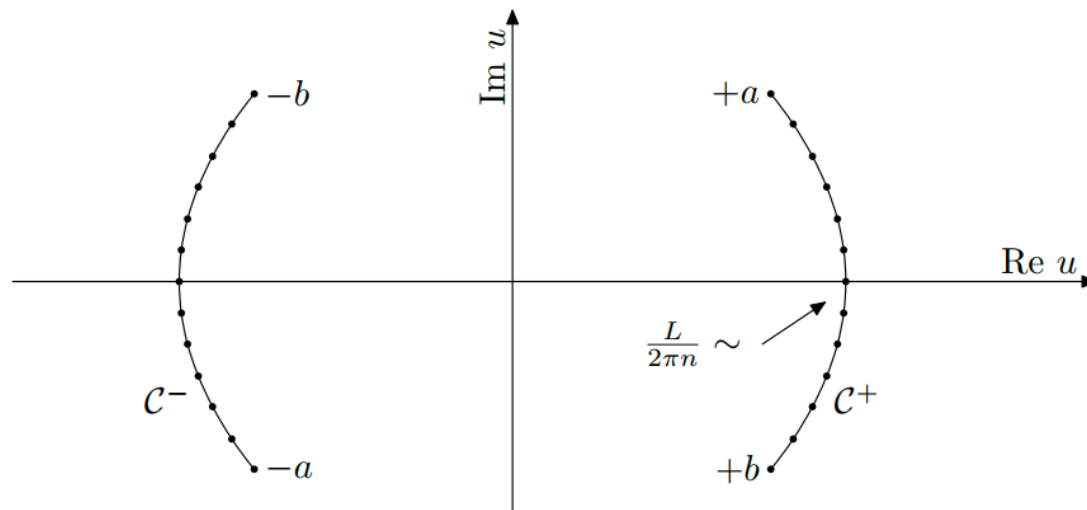
In integrability in AdS/CFT, for $L, N \gg 1$, with N/L fixed, solutions corresponds to strings in anti-de Sitter space

Specific solutions

Example 2. Semi-classical solution

In integrability in AdS/CFT, for $L, N \gg 1$, with N/L fixed, solutions corresponds to strings in anti-de Sitter space

A two-cut solution [spinning string in AdS]



[Figure from *JHEP* 09 (2003) 010]

[Beisert, Minahan, Staudacher, Zarembo 2003]

All solutions

Why consider all solutions ?

Completeness

A fundamental question of Bethe ansatz : does it give too little, too many or the right number of solutions ?

Applications

In many applications we need to sum over complete states, when no state clearly dominant, we need to sum over all solutions.

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Basic fact

BAE of XXX chain has ***too many*** solutions, some are ***non-physical***

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Applications

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Non-physical solutions

Solutions of BAE that do not correspond to eigenstates

Non-physical solutions

Repeated roots

$$\{u, u, u_1, u_2, \dots, u_{N-2}\}$$

Non-physical solution for **spin-1/2 XXX**

Non-physical solutions

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$$\{u, u, u_1, u_2, \dots, u_{N-2}\}$$

Non-physical solution for **spin-1/2 XXX**

!! For **spin- s XXX chain**, there exists such **physical solutions**

Non-physical solutions

Repeated roots

$$\{\textcolor{red}{u}, \textcolor{red}{u}, u_1, u_2, \dots, u_{N-2}\}$$

Non-physical solution for **spin-1/2 XXX**

!! For **spin- s XXX chain**, there exists such **physical solutions**

Singular solutions

$$\{\textcolor{red}{i/2}, \textcolor{red}{-i/2}, u_1, u_2, \dots, u_{N-2}\}$$

$$E_N \propto \sum_{j=1}^N \frac{1}{u_j^2 + 1/4} \quad \longrightarrow \quad \text{Singularity}$$

Careful **regularizations** are needed !

Non-physical solutions

Repeated roots

$$\{u, u, u_1, u_2, \dots, u_{N-2}\}$$

Non-physical solution for **spin-1/2 XXX**

!! For **spin- s XXX chain**, there exists such **physical solutions**

Physical conditions

$$\{i/2, -i/2, u_1, u_2, \dots, u_{N-2}\}$$

$$\left[- \prod_{j=1}^{N-2} \left(\frac{u_j + i/2}{u_j - i/2} \right) \right]^L = 1$$



Physical solution

Baxter's TQ -relation

The TQ -relation

$$T(u)Q(u) = a(u)Q(u - i) + d(u)Q(u + i)$$

$$a(u) = \left(u + \frac{i}{2}\right)^L, \quad d(u) = \left(u - \frac{i}{2}\right)^L$$

$T(u)$ and $Q(u)$ are **polynomials in u**

$$Q(u) = \prod_{j=1}^N (u - u_j)$$

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$T(u)$ and $Q(u)$ are **polynomials in u**

$$Q(u) = \prod_{j=1}^N (u - u_j)$$

Solve TQ -relation

$$T(u) = \sum_{k=0}^L t_k u^k, \quad Q(u) = \sum_{k=1}^N c_k u^k$$

Plug in TQ -relation and solve for $\{t_k, c_k\}$

TQ and Completeness

The TQ -relation

2nd order difference equation

$$T(u)Q(u) = a(u)Q(u - i) + d(u)Q(u + i)$$

Completeness

The TQ -relation has **another solution** $P(u)$

Theorem

The solution of TQ -relation for **both** $Q(u)$ and $P(u)$ **being polynomials** give the complete spectrum of the XXX spin chain.

[Granet and Jacobsen 2019]

[Mukhin, Tarasov, Varchenko 2009]

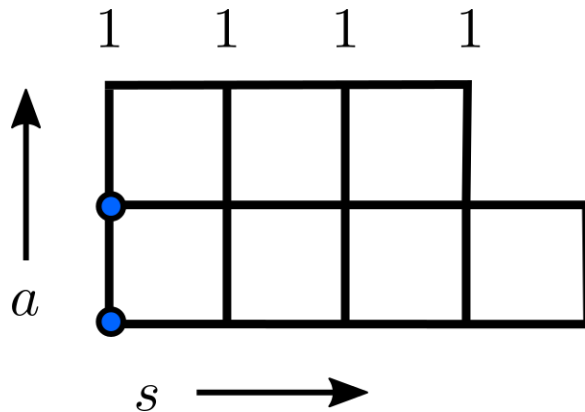
Rational Q-system

Start from a set of BAE

[Marboe and Volin 2016]

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} \right)^L = \prod_{\substack{k=1 \\ k \neq j}}^N \frac{u_j - u_k + i}{u_j - u_k - i}, \quad j = 1, \dots, N$$

Young diagram



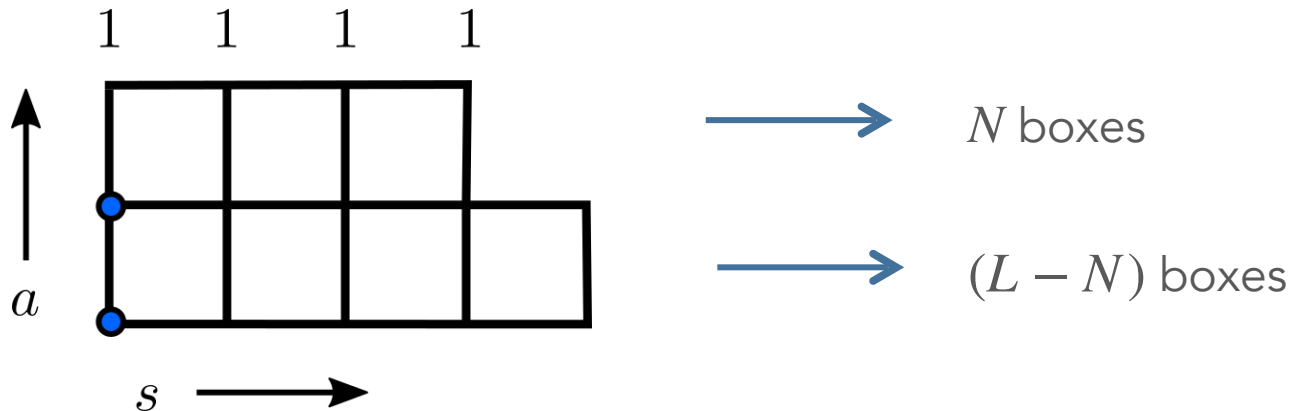
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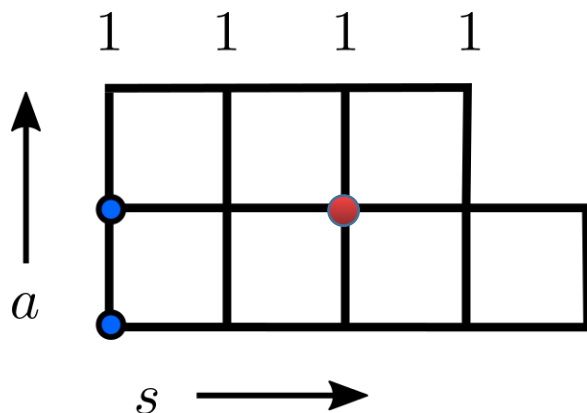
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Young diagram



A Q -function at each node

$$Q_{a,s}(u)$$

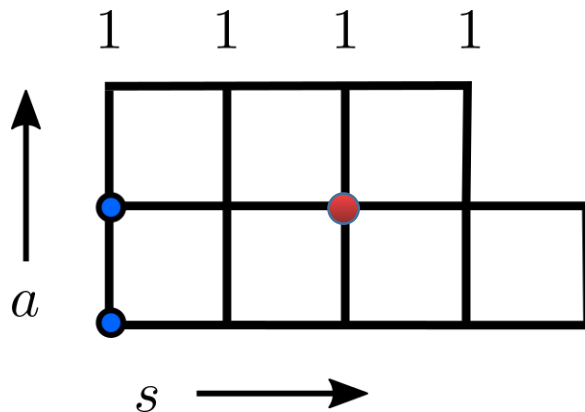
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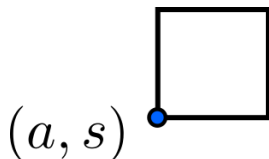
Young diagram



A Q -function at each node

$$Q_{a,s}(u)$$

QQ-relation at each box



$$Q_{a+1,s} Q_{a,s+1} = Q_{a+1,s+1}^- Q_{a,s}^+ - Q_{a+1,s+1}^+ Q_{a,s}^-$$

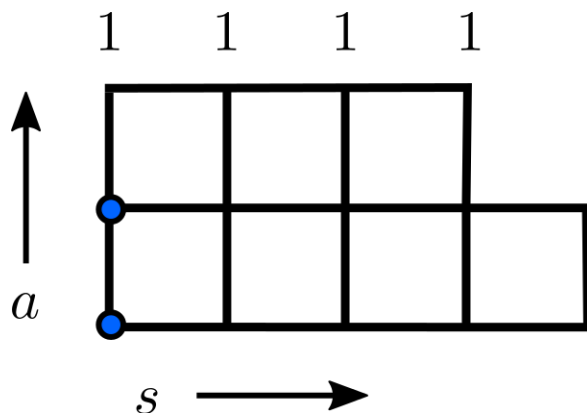
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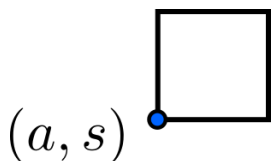
Young diagram



A Q -function at each node

$$Q_{a,s}^{\pm} \equiv Q_{a,s}(u \pm \frac{i}{2})$$

QQ-relation at each box



$$Q_{a+1,s} Q_{a,s+1} = Q_{a+1,s+1}^- Q_{a,s}^+ - Q_{a+1,s+1}^+ Q_{a,s}^-$$

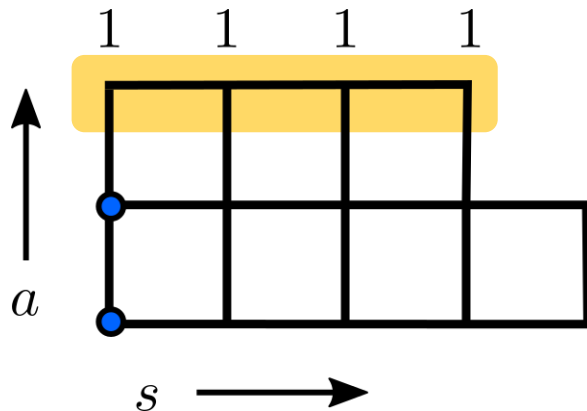
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Young diagram



Q -functions on upper
boundary are trivial

Boundary conditions 1

$$Q_{2,s}(u) = 1$$

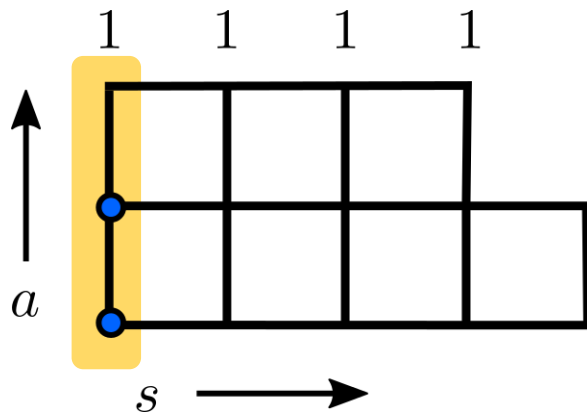
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Young tableaux



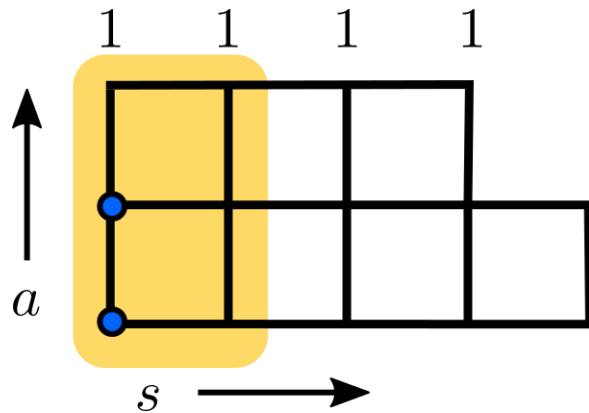
Q -functions on left boundary
are partially fixed

Boundary conditions 2

$$Q_{0,0}(u) = u^L, \quad Q_{1,0}(u) = \prod_{j=1}^N (u - u_j)$$

Rational Q-system

From rational Q -system to BAE



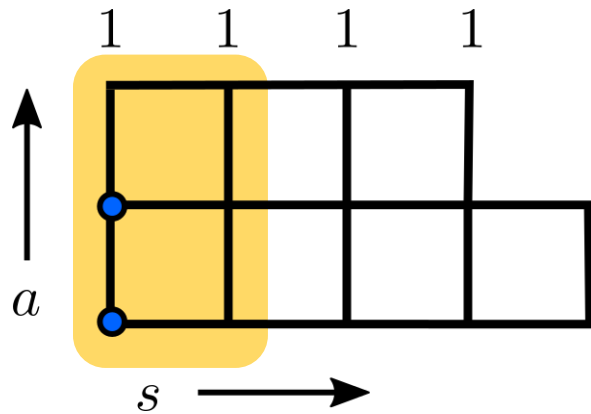
Focus on the first column

$$Q_{1,1} = Q_{1,0}^+ - Q_{1,0}^-$$

$$Q_{1,0}Q_{0,1} = Q_{1,1}^-Q_{0,0}^+ - Q_{1,1}^+Q_{0,0}^-$$

Rational Q-system

From rational Q -system to BAE



Focus on the first column

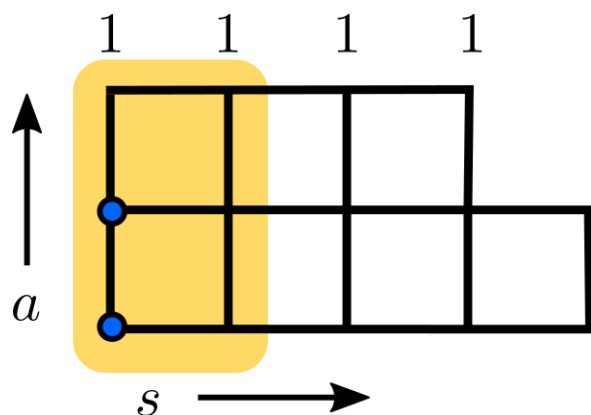
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$$Q_{1,1}^-(u_k)Q_{0,0}^+(u_k) = Q_{1,1}^+(u_k)Q_{0,0}^-(u_k)$$

Rational Q-system

From rational Q -system to BAE



Focus on the first column

$$Q_{1,1} = Q_{1,0}^+ - Q_{1,0}^-$$

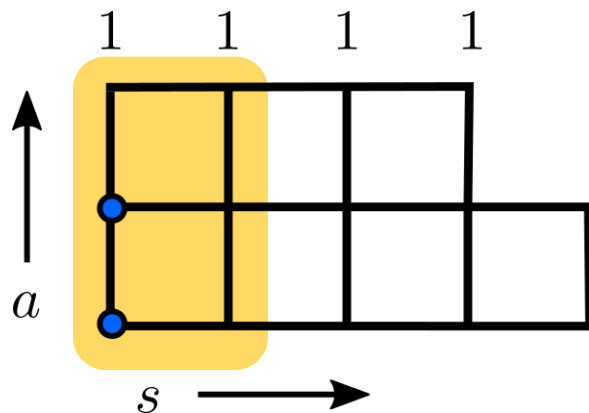
$$Q_{1,0}Q_{0,1} = Q_{1,1}^- Q_{0,0}^+ - Q_{1,1}^+ Q_{0,0}^-$$

$$\begin{aligned} Q_{1,1}^+(u_k) &= Q_{1,0}^{++}(u_k), \\ Q_{1,1}^-(u_k) &= -Q_{1,0}^{--}(u_k) \end{aligned}$$

$$Q_{1,1}^-(u_k)Q_{0,0}^+(u_k) = Q_{1,1}^+(u_k)Q_{0,0}^-(u_k)$$

Rational Q-system

From rational Q -system to BAE



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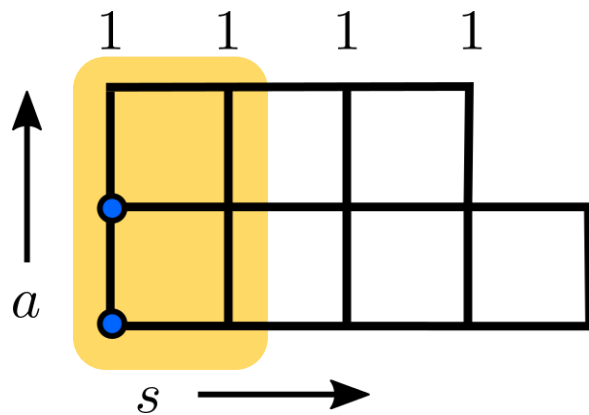
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$$\frac{Q_{0,0}^+(u_k)}{Q_{0,0}^-(u_k)} \frac{Q_{1,0}^{--}(u_k)}{Q_{1,0}^{++}(u_k)} = -1$$

BAE !

Rational Q-system

From rational Q -system to BAE



Key requirement:

All Q -functions on the Young diagram are **polynomials**

1. Parametrization

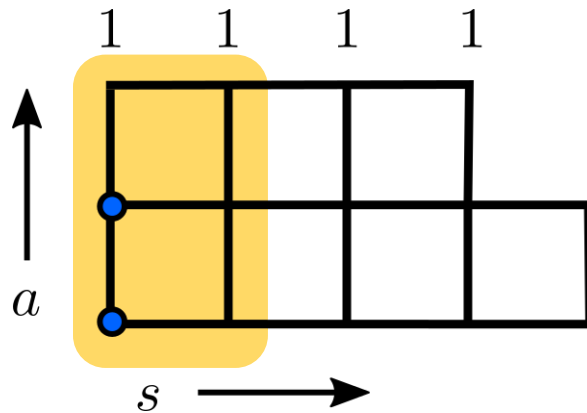
$$Q_{1,0}(u) = u^N + \sum_{j=0}^{N-1} c_j u^j$$

2. Solving first row

$$Q_{1,s}(u) = D^s Q_{1,0}(u), \quad Df(u) = f(u + \tfrac{i}{2}) - f(u - \tfrac{i}{2})$$

Rational Q-system

Solving rational Q -system



Key requirement:

All Q -functions on the Young diagram are **polynomials**

3. Solving the second row

$$Q_{0,s+1} = \frac{Q_{1,s+1}^- Q_{0,s}^+ - Q_{1,s+1}^+ Q_{0,s}^-}{Q_{1,s}}$$

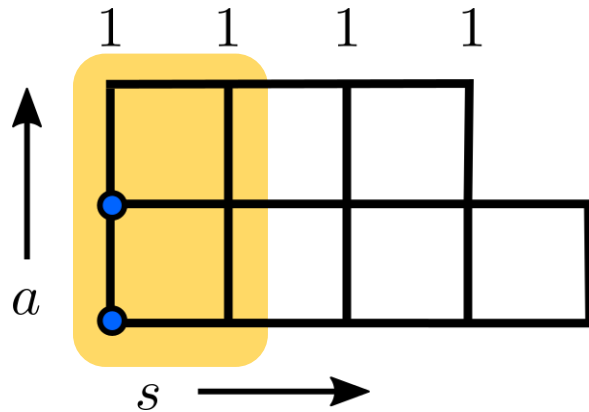
Require the **remainder** of this quotient to **vanish**.

Obtain **zero remainder conditions**, algebraic equations for $\{c_j\}$

By far **most efficient**, only **physical solutions**

Rational Q-system

Solving rational Q -system



Key requirement:

All Q -functions on the Young diagram are **polynomials**

3. Solving the second row

$$Q_{0,s+1} = \frac{Q_{1,s+1}^- Q_{0,s}^+ - Q_{1,s+1}^+ Q_{0,s}^-}{Q_{1,s}}$$

Remark

1. Gives all and only physical solutions;
2. Much more efficient

q -deformation

Generic v.s. Root of unity

XXZ spin chain

The model

$$H_{\text{XXZ}} = \sum_{j=1}^L \left(\sigma_j^- \sigma_{j+1}^+ + \sigma_j^+ \sigma_{j+1}^- + \Delta \sigma_j^z \sigma_{j+1}^z \right)$$

The **anisotropy** parameter

$$\Delta = \frac{1}{2} (q + q^{-1}) = \cosh \eta$$

Root of unity

$$\eta = i\pi \frac{\ell_1}{\ell_2}, \quad q^{\ell_2} = \varepsilon = \pm 1$$

Otherwise it is called **generic**

XXZ at generic q

The Bethe ansatz equation

$$\left(\frac{\sinh(u_j + \eta/2)}{\sinh(u_j - \eta/2)} \right)^L = \prod_{\substack{k=1 \\ k \neq j}}^N \frac{\sinh(u_j - u_k + \eta)}{\sinh(u_j - u_k - \eta)}, \quad j = 1, \dots, N$$

XXZ at generic q

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The TQ -relation

$$T(u)Q(u) = a(u)Q(u - \eta) + d(u)Q(u + \eta)$$

$$a(u) = \sinh^L(u + \frac{\eta}{2}), \quad d(u) = \sinh^L(u - \frac{\eta}{2})$$

$$Q(u) = \prod_{j=1}^N \sinh(u - u_j)$$

Multiplicative variables

$$t = e^u, \quad t_j = e^{u_j}, \quad q = e^\eta$$

XXZ at generic q

The Bethe ansatz equation

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The TQ -relation

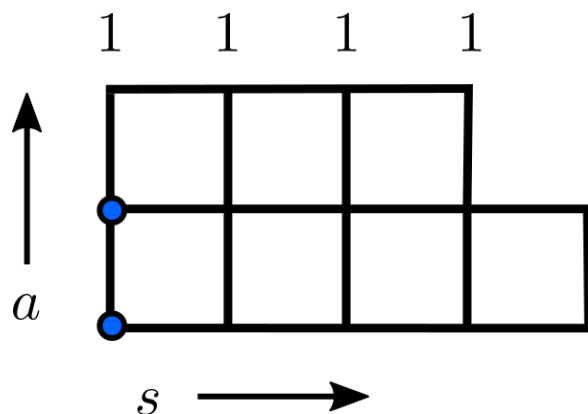
$$T(u)Q(u) = a(u)Q(u - \eta) + d(u)Q(u + \eta)$$

$$a(t) \propto (tq^{1/2} - t^{-1}q^{-1/2})^L, \quad d(t) \propto (tq^{-1/2} - t^{-1}q^{1/2})^L$$

$$Q(t) \propto \prod_{j=1}^N (t^2 - t_j^2)$$

Q-system at generic q

The QQ -relation



$$Q_{0,0}(t) = (t - t^{-1})^L$$

$$Q_{1,0}(t) = \prod_{j=1}^N (t - t^{-1} t_j^2)$$

$$f^{\pm}(t) \equiv f(tq^{\pm 1/2})$$

$$Q_{a+1,s}(t)Q_{a,s+1}(t) \propto Q_{a+1,s+1}^+(t)Q_{a,s}^-(t) - Q_{a+1,s+1}^-(t)Q_{a,s}^+(t)$$

Analytic property

Require that all $Q_{a,s}(t)$ are **Laurent polynomials** in t

Root of unity

The problem

Solve the TQ -relation like generic q , find **infinitely many solutions**....

Root of unity

The problem

Solve the TQ -relation like generic q , find **infinitely many solutions**....

Fabricius-McCoy string

$$\tilde{u}_n = u_0 + \frac{i\pi(n-1)}{\ell_2}, \quad u_0 \in \mathbb{C}, \quad n = 1, 2, \dots, \ell_2$$

‘Transparent’ property

$$\prod_{n=1}^{\ell_2} \frac{\sinh(u - u_n + \eta)}{\sinh(u - u_n - \eta)} = 1$$

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If

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[Fabricsius and McCoy 2001]

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[Fabricsius and McCoy 2001]

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If

$$\{u_1, u_2, \dots, u_N\}$$

Is a **solution of BAE**

Root of unity

The problem

Solve the TQ -relation like generic q , find **infinitely many solutions**....

Fabricsius-McCoy string

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‘Transparent’ property

$$\prod_{n=1}^{\ell_2} \frac{\sinh(u - u_n + \eta)}{\sinh(u - u_n - \eta)} = 1$$

Then

$$\{u_1, u_2, \dots, u_N, \tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_{\ell_2}\}$$

Is **also** a solution of BAE, the center u_0 **can be arbitrary**

Root of unity

Structure of Hilbert space

$$\{u_1, u_2, \dots, u_N\}$$

Root of unity

Structure of Hilbert space

$$\{u_1, u_2, \dots, u_N\}$$

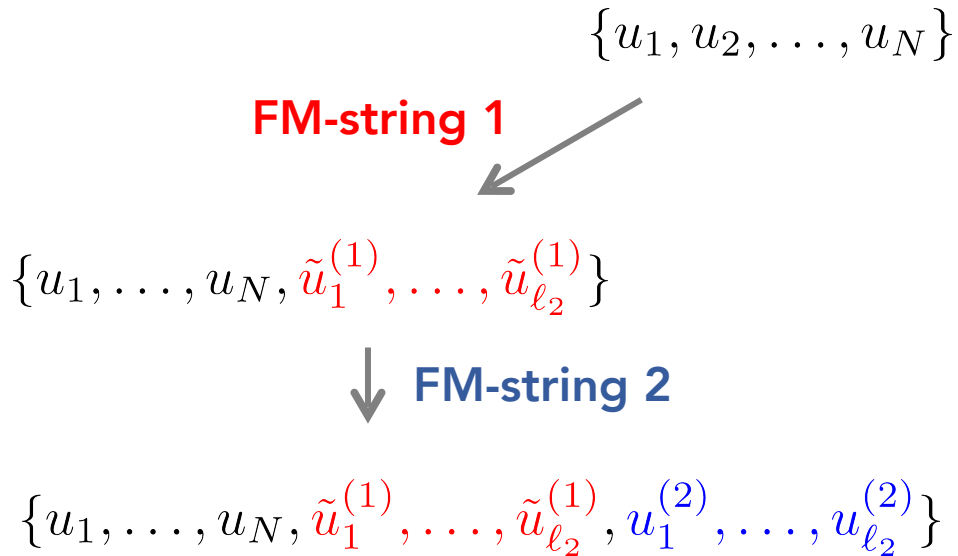
FM-string 1



$$\{u_1, \dots, u_N, \tilde{u}_1^{(1)}, \dots, \tilde{u}_{\ell_2}^{(1)}\}$$

Root of unity

Structure of Hilbert space



Root of unity

Structure of Hilbert space

$$\{u_1, u_2, \dots, u_N\}$$

FM-string 1



$$\{u_1, \dots, u_N, \tilde{u}_1^{(1)}, \dots, \tilde{u}_{\ell_2}^{(1)}\}$$



FM-string 2

$$\{u_1, \dots, u_N, \tilde{u}_1^{(1)}, \dots, \tilde{u}_{\ell_2}^{(1)}, u_1^{(2)}, \dots, u_{\ell_2}^{(2)}\}$$

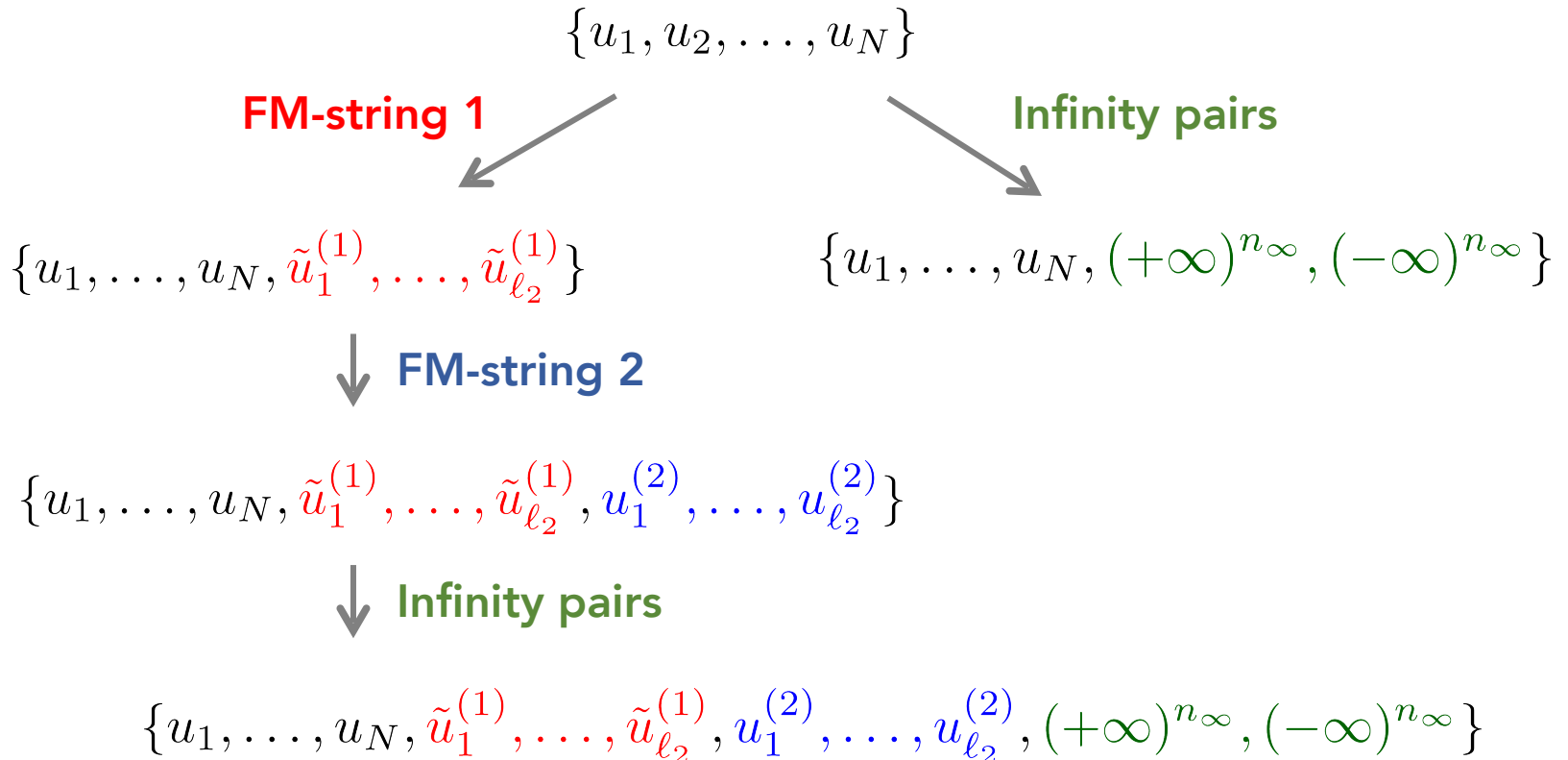


Infinity pairs

$$\{u_1, \dots, u_N, \tilde{u}_1^{(1)}, \dots, \tilde{u}_{\ell_2}^{(1)}, u_1^{(2)}, \dots, u_{\ell_2}^{(2)}, (+\infty)^{n_\infty}, (-\infty)^{n_\infty}\}$$

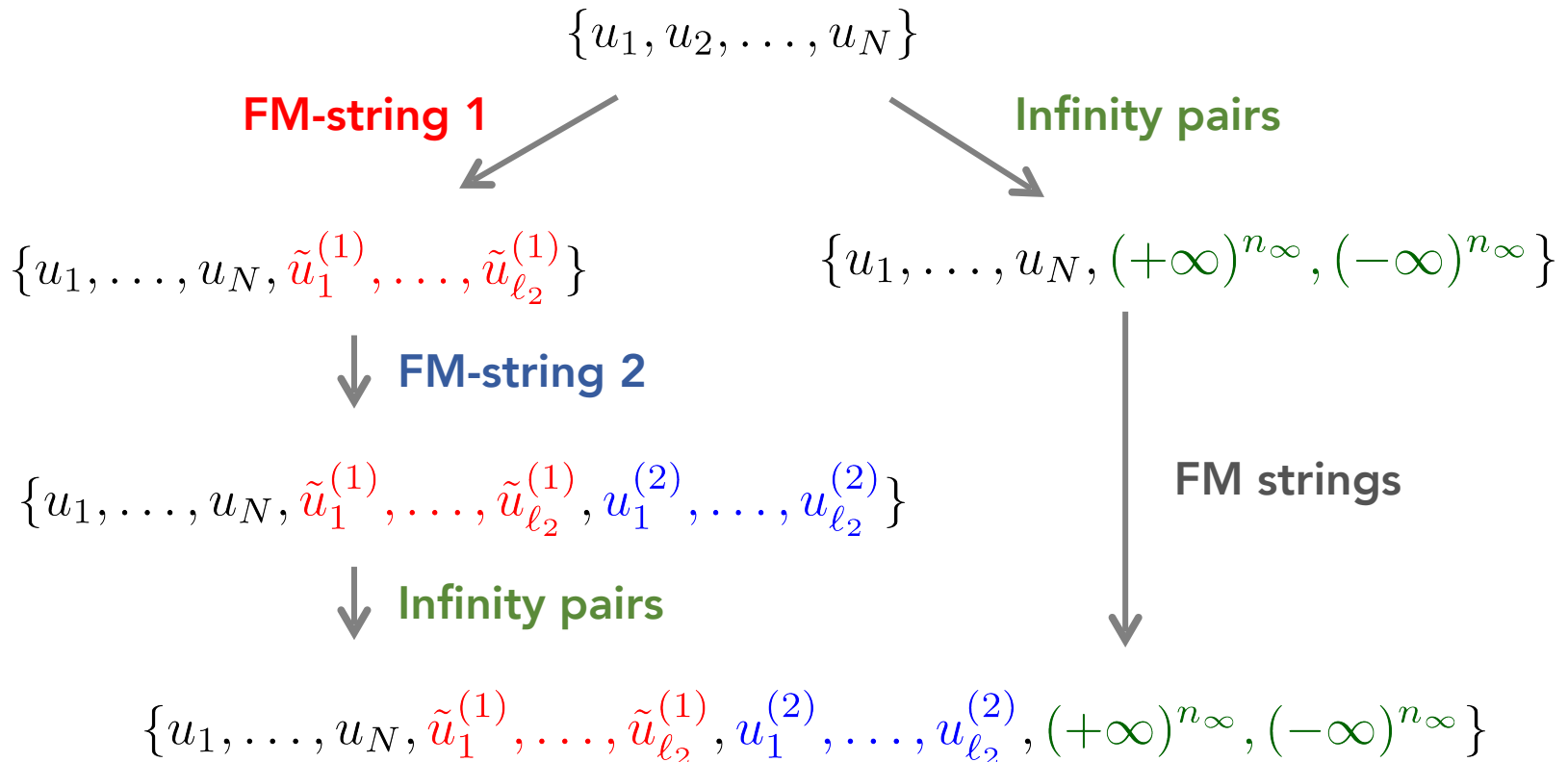
Root of unity

Structure of Hilbert space



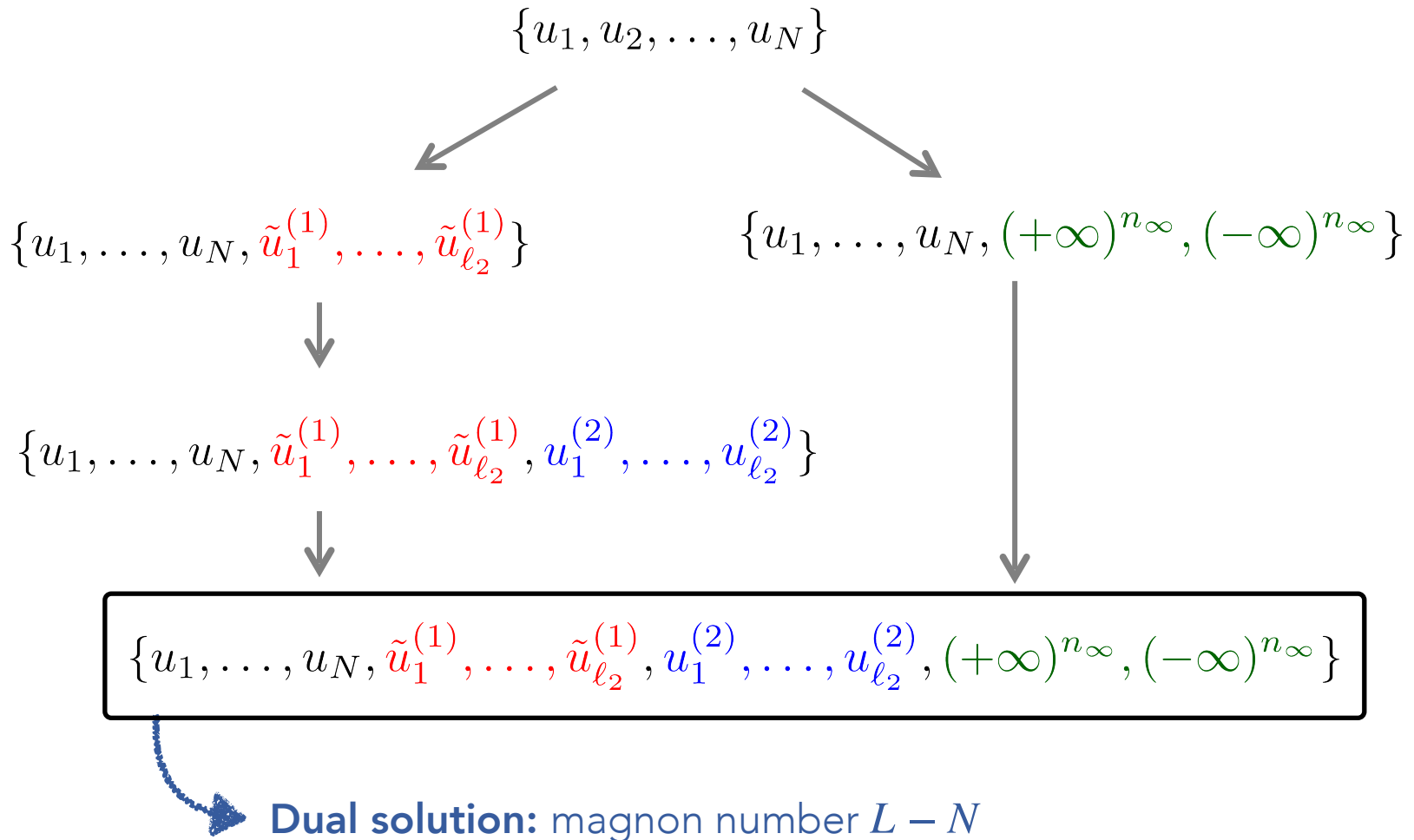
Root of unity

Structure of Hilbert space



Root of unity

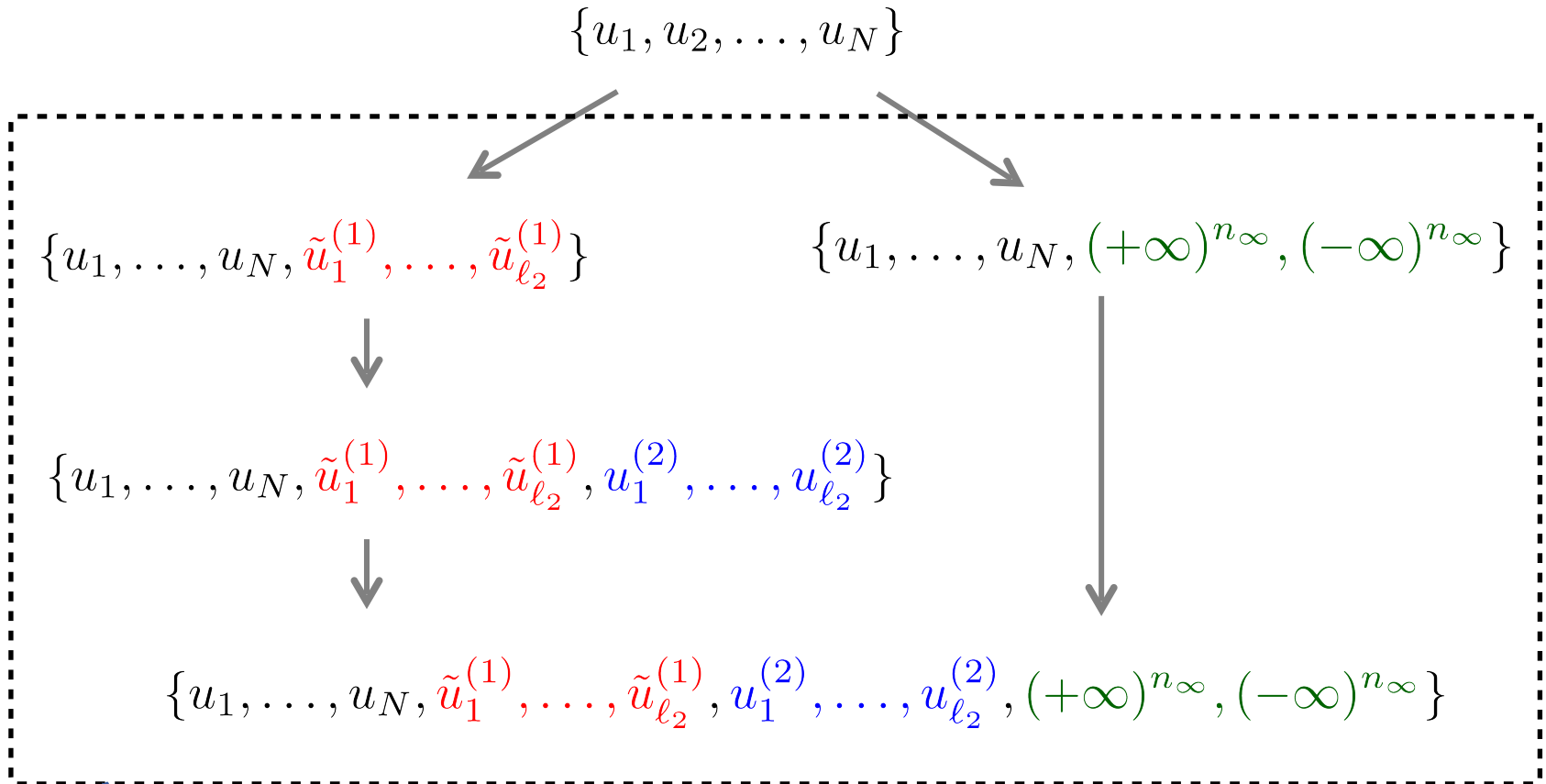
Structure of Hilbert space



[Miao, Lamers, Pasquier 2021]

Root of unity

Structure of Hilbert space



Descendent state : Same energy and conserved charges !

Root of unity

Step 1. Find all **primitive** states

- Finite solutions

$$\{u_1, u_2, \dots, u_N\}$$

- Finite solutions + **positive infinities**

$$\{u_1, u_2, \dots, u_N, \underbrace{\infty, \dots, \infty}_{n_+}\}$$

- Finite solutions + **negative infinities**

$$\{u_1, u_2, \dots, u_N, \underbrace{-\infty, \dots, -\infty}_{n_-}\}$$

Step 2. For each *primitive state*, find all descendants

- Primitive solution + **FM strings** + **infinity pairs**

Root of unity

The *constrained* Q-system

[Hou, Jiang and Miao 2023]

Idea: impose additional constraints to get rid of FM-strings

A solution contain **FM-string** $Q(t) = Q^{\text{FM}}(t)\tilde{Q}(t)$

$$Q(t) = \prod_{j=1}^{\ell_2} \left(t - \frac{(t_j^{\text{FM}})^2}{t} \right) \prod_{k=\ell_2+1}^N \left(t - \frac{t_k^2}{t} \right) = t^N + \sum_{k=1}^{N-1} c_k t^{2k-N} + \frac{t^{-N}}{c_0}$$

A system of equations involving $\{c_l\}, \{t_k^{\text{FM}}\}, \{t_j\}$

Root of unity

The *constrained* Q-system

[Hou, Jiang and Miao 2023]

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A system of equations involving $\{c_l\}, \{t_k^{\text{FM}}\}, \{t_j\}$



Eliminate $\{t_k^{\text{FM}}\}, \{t_j\}$

$$R_k(c_0, c_1, \dots, c_{N-1}) = 0, \quad k = 1, \dots, N_c$$

Root of unity

The *constrained* Q-system

[Hou, Jiang and Miao 2023]

Idea: impose additional constraints to get rid of FM-strings

A solution contain **FM-string** $Q(t) = Q^{\text{FM}}(t)\tilde{Q}(t)$

$$Q(t) = \prod_{j=1}^{\ell_2} \left(t - \frac{(t_j^{\text{FM}})^2}{t} \right) \prod_{k=\ell_2+1}^N \left(t - \frac{t_k^2}{t} \right) = t^N + \sum_{k=1}^{N-1} c_k t^{2k-N} + \frac{t^{-N}}{c_0}$$

Impose the condition

$$1 + w (|R_1|^2 + |R_2|^2 + \cdots |R_{N_c}|^2) = 0$$

Root of unity

Example

$$L = 8, \quad M = 4, \quad \eta = \frac{i\pi}{3}$$

FM-strings

$$\tilde{u}_1 = u_0 - \frac{2\pi i}{3}, \quad \tilde{u}_2 = u_0, \quad \tilde{u}_3 = u_0 + \frac{2\pi i}{3}$$

This leads to

$$c_0 = \frac{e^{-6u_0}}{t_4^2}, \quad c_1 = -e^{6u_0}, \quad c_2 = 0, \quad c_3 = -t_4^2$$

Obtain a set of equations involving $\{c_l\}$

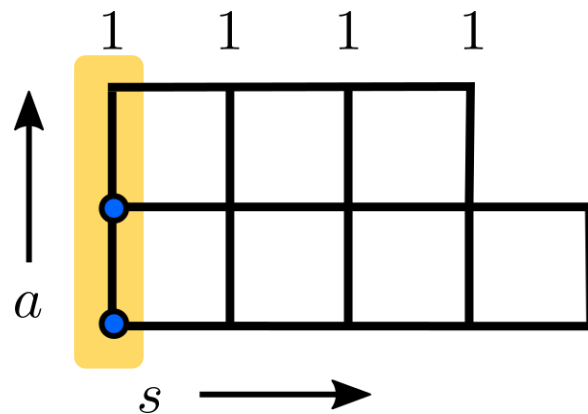
$$c_2 = 0, \quad c_0 c_1 c_3 - 1 = 0$$

Impose constraint

$$1 + w [c_2^2 + (c_0 c_1 c_3 - 1)^2] = 0$$

Root of unity

Constrained Q-system



$$Q_{0,0}(t) = (t - t^{-1})^L$$

$$Q_{1,0}(t) = \prod_{j=1}^N (t - t^{-1} t_j^2)$$

$$Q_{a+1,s}(t)Q_{a,s+1}(t) \propto Q_{a+1,s+1}^+(t)Q_{a,s}^-(t) - Q_{a+1,s+1}^-(t)Q_{a,s}^+(t)$$

+ constraints that eliminate FM-strings

$$1 + w \left(|R_1|^2 + |R_2|^2 + \cdots |R_{N_c}|^2 \right) = 0$$

- This gives all primitive solutions
- Similar strategy for primitive solutions **with infinities**

Root of unity

Descendant states

The center of FM-string can be fixed

A factorization relation

$$\mathbf{T}_s(u) = \mathbf{Q}\left(u + \frac{2s+1}{2}\eta\right) \tilde{\mathbf{Q}}\left(u - \frac{2s+1}{2}\eta\right)$$

Eigenvalue equation

$$T_s(u) = Q\left(u + \frac{2s+1}{2}\eta\right) \tilde{Q}\left(u - \frac{2s+1}{2}\eta\right)$$

Can prove that

$$\tilde{Q}(u) = Q(u)F(u)$$

Zeros of $F(u)$ contain **FM-strings** and **infinity pairs**

[Miao, Lamers, Pasquier 2021]

Root of unity

Fusion relation

$$\mathbf{T}_{1/2}^{[2s+1]}(u) \mathbf{T}_s(u) = T_0^{[2s+2]}(u) \mathbf{T}_{s-1/2}^{[-1]}(u) + T_0^{[2s]}(u) \mathbf{T}_{s-1/2}^{[+1]}(u)$$

Where $f^{[a]}(u) \equiv f\left(u + a\frac{\eta}{2}\right)$

Root of unity

Fusion relation

$$\mathbf{T}_{1/2}^{[2s+1]}(u) \mathbf{T}_s(u) = T_0^{[2s+2]}(u) \mathbf{T}_{s-1/2}^{[-1]}(u) + T_0^{[2s]}(u) \mathbf{T}_{s-1/2}^{[+1]}(u)$$

Where $f^{[a]}(u) \equiv f\left(u + a\frac{\eta}{2}\right)$



General solution

$$T_s(u) = Q^{[2s+1]}(u) Q^{[-2s-1]} \sum_{k=0}^{2s} \frac{T_0^{[2(k-s)]}(u)}{Q^{[2(k-s)-1]}(u) Q^{[2(k-s)+1]}(u)}$$

Root of unity

Fusion relation

$$\mathbf{T}_{1/2}^{[2s+1]}(u) \mathbf{T}_s(u) = T_0^{[2s+2]}(u) \mathbf{T}_{s-1/2}^{[-1]}(u) + T_0^{[2s]}(u) \mathbf{T}_{s-1/2}^{[+1]}(u)$$

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General solution

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We find

$$F(u) = \sum_{k=0}^{2s} \frac{T_0^{[2k+1]}(u)}{Q^{[2k]}(u) Q^{[2k+2]}(u)}$$

Root of unity

Example

$$L = 8, \quad M = 1, \quad \eta = \frac{2\pi i}{3}$$

Primitive solution : $t_1 = i$

$$Q(t) = t + t^{-1}$$

Find that

$$F(t) = 3(t^6 - 29 + t^{-6})$$

Center of the FM-strings

$$t_1^{\text{FM}} = \left(\frac{1}{2}(29 + 3\sqrt{93}) \right)^{1/6}, \quad t_2^{\text{FM}} = \left(\frac{1}{2}(29 - 3\sqrt{93}) \right)^{1/6}$$

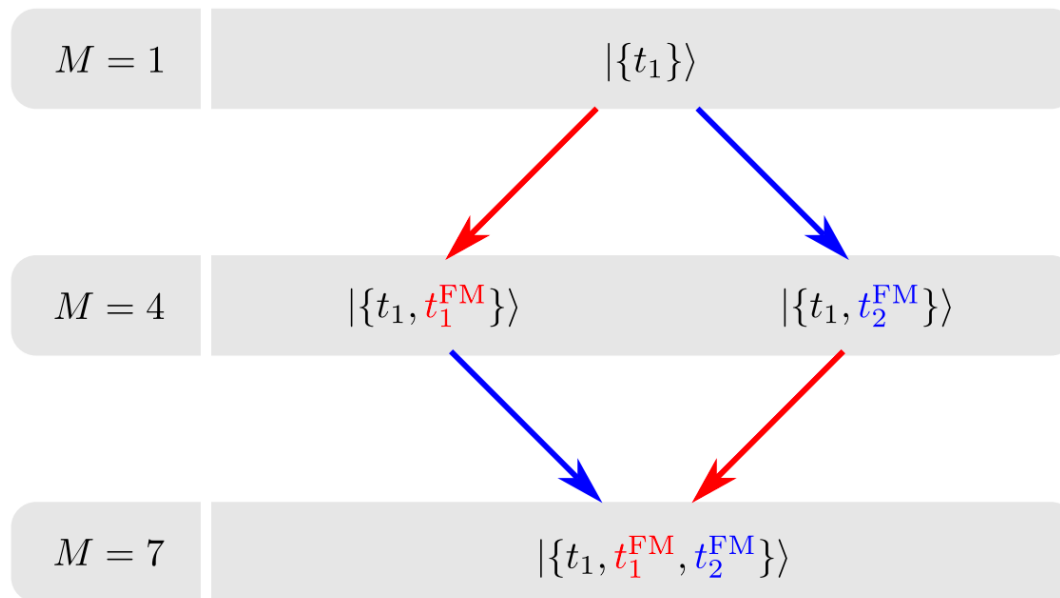
Root of unity

Example

$$L = 8, \quad M = 1, \quad \eta = \frac{2\pi i}{3}$$

Primitive solution : $t_1 = i$

Structure of the multiplet



$U(1)$ -Breaking

Twists and Boundaries

A new perspective

Question

How to derive the Q -system ?

A new perspective

Question

How to derive the Q -system ?

A new perspective

$$Q\text{-system} = TQ\text{-relation} + \text{fusion} (T\text{-system})$$

A new perspective

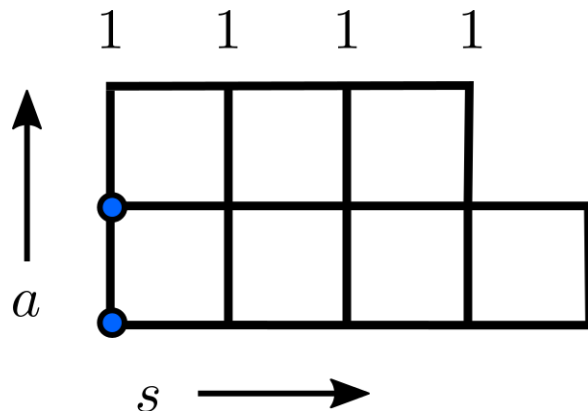
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How to derive the Q -system ?

A new perspective

$$Q\text{-system} = TQ\text{-relation} + \text{fusion} (T\text{-system})$$

The minimum Q -system



A new perspective

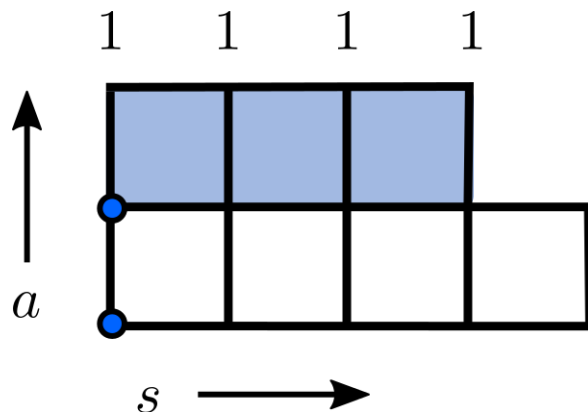
Question

How to derive the Q -system ?

A new perspective

$$Q\text{-system} = TQ\text{-relation} + \text{fusion (} T\text{-system)}$$

The minimum Q -system



$$Q_{1,n} = Q_{1,n-1}^+ - Q_{1,n-1}^-$$

A new perspective

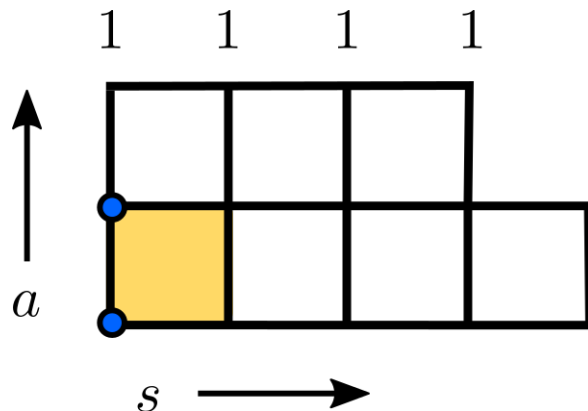
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How to derive the Q -system ?

A new perspective

$$Q\text{-system} = TQ\text{-relation} + \text{fusion (} T\text{-system)}$$

The minimum Q -system



$$Q_{1,n} = Q_{1,n-1}^+ - Q_{1,n-1}^-$$

$$Q_{1,0}Q_{0,1} = Q_{1,1}^- Q_{0,0}^+ - Q_{1,1}^+ Q_{0,0}^-$$

A new perspective

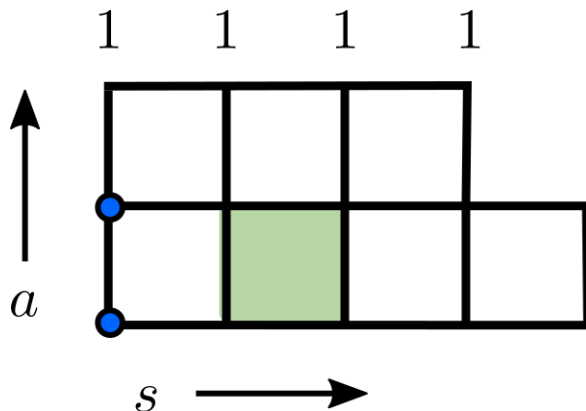
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A new perspective

$$Q\text{-system} = TQ\text{-relation} + \text{fusion (} T\text{-system)}$$

The minimum Q -system



$$Q_{1,n} = Q_{1,n-1}^+ - Q_{1,n-1}^-$$

$$Q_{1,0}Q_{0,1} = Q_{1,1}^- Q_{0,0}^+ - Q_{1,1}^+ Q_{0,0}^-$$

$$Q_{1,1}Q_{0,2} = Q_{1,2}^+ Q_{0,1}^- - Q_{1,2}^- Q_{0,1}^+$$

A new perspective

Q -system

$$Q_{1,n} = Q_{1,n-1}^+ - Q_{1,n-1}^-$$

$$Q_{1,0}Q_{0,1} = Q_{1,1}^- Q_{0,0}^+ - Q_{1,1}^+ Q_{0,0}^-$$

$$Q_{1,1}Q_{0,2} = Q_{1,2}^+ Q_{0,1}^- - Q_{1,2}^- Q_{0,1}^+$$

TQ +Fusion relation

$$T_{1/2}Q = T_0^+ Q^{[-2]} + T_0^- Q^{[+2]}$$

$$T_{1/2}^{[+2]} T_{1/2} = T_0^{[+3]} T_0^- + T_0^+ T_1^+$$

A new perspective

Q -system

$$Q_{1,n} = Q_{1,n-1}^+ - Q_{1,n-1}^-$$

$$Q_{1,0}Q_{0,1} = Q_{1,1}^- Q_{0,0}^+ - Q_{1,1}^+ Q_{0,0}^-$$

$$Q_{1,1}Q_{0,2} = Q_{1,2}^+ Q_{0,1}^- - Q_{1,2}^- Q_{0,1}^+$$

Equivalent !

TQ +Fusion relation

$$T_{1/2}Q = T_0^+ Q^{[-2]} + T_0^- Q^{[+2]}$$

$$T_{1/2}^{[+2]} T_{1/2} = T_0^{[+3]} T_0^- + T_0^+ T_1^+$$

$$Q(u) = Q_{1,0}(u), \quad Q_{1,n} = Q_{1,n-1}^{[+1]} - Q_{1,n-1}^{[-1]}$$

$$Q_{0,1} = T_0^{[+1]} + T_0^{[-1]} - T_{1/2}$$

$$Q_{0,2} = T_1 - 2 \left(T_{1/2}^{[+1]} + T_{1/2}^{[-1]} \right) + \left(T_0^{[+2]} + T_0^{[-2]} + 3T_0 \right)$$

[Jiang, Liu, Miao, Tan, to appear]

A new perspective

Key idea

Use TQ +Fusion system, derive Q -system

A new perspective

Key idea

Use TQ +Fusion system, derive Q -system

Good news

- TQ -relation is known for a lot of integrable models
- Fusion relation can be derived from R -matrix
- These relations also exist for $U(1)$ -breaking models

A new perspective

Key idea

Use TQ +Fusion system, derive Q -system

Good news

- TQ -relation is known for a lot of integrable models
- Fusion relation can be derived from R -matrix
- These relations also exist for $U(1)$ -breaking models

Two $U(1)$ -breaking models

- XXZ/XXX spin chain with *non-diagonal twist*
- XXZ/XXX spin chain with *un-parallel boundary conditions*

Twisted spin chain

Anti-diagonal twist

$$H = \sum_{j=1}^{L-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z) + \sigma_L^x \sigma_1^x - \sigma_L^y \sigma_1^y - \Delta \sigma_L^z \sigma_1^z$$

Twisted spin chain

Anti-diagonal twist

$$H = \sum_{j=1}^{L-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z) + \sigma_L^x \sigma_1^x - \sigma_L^y \sigma_1^y - \Delta \sigma_L^z \sigma_1^z$$

Inhomogeneous Q-system

$$q^{-1/2} T_{1/2} Q = t T_0^- Q^{[+2]} - t^{-1} T_0^+ Q^{[-2]} - c(t) T_0^+ T_0^-$$


$$c(t) = q^{L/2} \left(\frac{t}{t_1 \cdots t_L} - \frac{t_1 \cdots t_L}{t} \right)$$

Twisted spin chain

Anti-diagonal twist

$$H = \sum_{j=1}^{L-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z) + \sigma_L^x \sigma_1^x - \sigma_L^y \sigma_1^y - \Delta \sigma_L^z \sigma_1^z$$

Inhomogeneous Q-system

$$q^{-1/2} T_{1/2} Q = t T_0^- Q^{[+2]} - t^{-1} T_0^+ Q^{[-2]} - c(t) T_0^+ T_0^-$$

Fusion relation

$$c(t) = q^{L/2} \left(\frac{t}{t_1 \cdots t_L} - \frac{t_1 \cdots t_L}{t} \right)$$

Similar but different to the periodic case

$$T_0 T_1 = T_{1/2}^+ T_{1/2}^- + T_0^{[+2]} T_0^{[-2]}$$

Twisted spin chain

Anti-diagonal twist

$$H = \sum_{j=1}^{L-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z) + \sigma_L^x \sigma_1^x - \sigma_L^y \sigma_1^y - \Delta \sigma_L^z \sigma_1^z$$

Identification rules

$$Q_{0,0} = T_0, \quad Q_{1,0} = Q$$

$$Q_{1,1} = tq^{-1/2}Q_{1,0}^+ - t^{-1}q^{1/2}Q_{1,0}^- \quad \text{1st row}$$

$$Q_{1,2} = tQ_{1,1}^+ - t^{-1}Q_{1,1}^-$$

2nd row

$$Q_{0,1} = q^{-1/2}T_{1/2} + tq^{-1}Q_{0,0}^+ - t^{-1}qQ_{0,0}^-$$

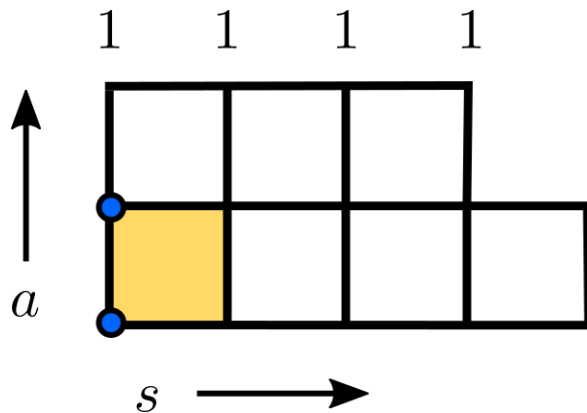
$$Q_{0,2} = T_1 + tq^{-1}T_{1/2}^+ - t^{-1}qT_{1/2}^- - q^{-2}Q_{0,0} + tq^{-1/2}Q_{0,1}^+ - t^{-1}q^{1/2}Q_{0,1}^-$$

Twisted spin chain

Anti-diagonal twist

$$H = \sum_{j=1}^{L-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z) + \sigma_L^x \sigma_1^x - \sigma_L^y \sigma_1^y - \Delta \sigma_L^z \sigma_1^z$$

Inhomogeneous Q -system



$$c(t) = q^{L/2} \left(\frac{t}{t_1 \cdots t_L} - \frac{t_1 \cdots t_L}{t} \right)$$

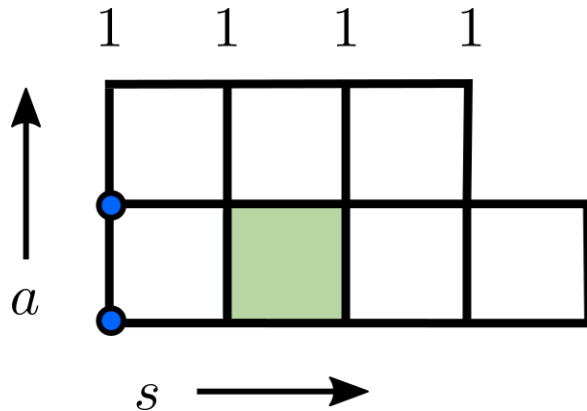
$$Q_{0,1} Q_{1,0} = Q_{0,0}^+ Q_{1,1}^- + Q_{0,0}^- Q_{1,1}^+ - c(t) Q_{0,0}^+ Q_{0,0}^-$$

Twisted spin chain

Anti-diagonal twist

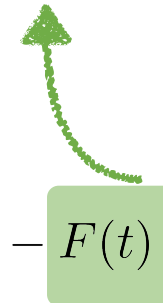
$$H = \sum_{j=1}^{L-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z) + \sigma_L^x \sigma_1^x - \sigma_L^y \sigma_1^y - \Delta \sigma_L^z \sigma_1^z$$

Inhomogeneous Q -system



$$F(t) = (tc(t))^+ Q_{0,1}^- T_0^{[+2]} - (t^{-1}c(t))^- Q_{0,1}^+ T_0^{[-2]}$$

$$Q_{0,2} Q_{1,1} = Q_{0,1}^+ Q_{1,2}^- + Q_{0,1}^- Q_{1,2}^+ - F(t)$$



Open spin chain

Open chain with magnetic field

$$H = \sum_{j=1}^{L-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z) + \vec{h}_1 \cdot \vec{\sigma}_1 + \vec{h}_L \cdot \vec{\sigma}_L$$

Magnetic fields

$$\vec{h}_1 = \frac{\sinh \eta}{\sinh \alpha_- \cosh \beta_-} (\cosh \theta_-, i \sinh \theta_-, \cosh \alpha_- \sinh \beta_-)$$

$$\vec{h}_L = \frac{\sinh \eta}{\sinh \alpha_+ \cosh \beta_+} (\cosh \theta_+, i \sinh \theta_+, -\cosh \alpha_+ \sinh \beta_+)$$

Generic parameters breaks the $U(1)$ symmetry

Open spin chain

Open chain with magnetic field

$$H = \sum_{j=1}^{L-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z) + \vec{h}_1 \cdot \vec{\sigma}_1 + \vec{h}_L \cdot \vec{\sigma}_L$$

Q -system

Define

$$Q_{0,0}(u) = \sinh^{2L}(u)$$

$$Q_{1,0}(u) = \prod_{j=1}^L \sinh(u - u_j) \sinh(u + u_j)$$

$$f(u) = g(-u) = \sinh(u - \alpha_+) \sinh(u - \alpha_-) \cosh(u - \beta_+) \cosh(u - \beta_-)$$

Open spin chain

Open chain with magnetic field

$$H = \sum_{j=1}^{L-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z) + \vec{h}_1 \cdot \vec{\sigma}_1 + \vec{h}_L \cdot \vec{\sigma}_L$$

Q -system

The QQ -relations

$$\sinh(2u)Q_{1,1} = g Q_{1,0}^+ - f Q_{1,0}^-$$

$$\sinh(2u)Q_{1,2} = g^- Q_{1,1}^+ - f^+ Q_{1,1}^-$$

$$\sinh(2u)Q_{0,1}Q_{1,0} = Q_{0,0}^+Q_{1,1}^- - Q_{0,0}^-Q_{1,1}^+ + \frac{x}{2} \sinh(2u)Q_{0,0}^+Q_{0,0}^-$$

$$\begin{aligned} \sinh(2u)Q_{0,2}Q_{1,1} &= Q_{0,1}^+Q_{1,2}^- - Q_{0,1}^-Q_{1,2}^+ \\ &+ \frac{x}{2} \left(g Q_{0,0}^{[+2]} Q_{0,1}^- - f Q_{0,0}^{[-2]} Q_{0,1}^+ \right) \end{aligned}$$

$$\begin{aligned} x &= \cosh(\alpha_+ + \beta_+ + \alpha_- + \beta_- + (L+1)\eta) \\ &\quad - \cosh(\theta_- - \theta_+) \end{aligned}$$

Conclusions

Solving BAE is an important step in many applications of Bethe ansatz

If we want to find all solutions, we can use the **rational Q-system**, which yields the complete solution

The rational Q-system can be generalized to **q -deformed case**, both for generic q and q at root of unity

The Q-system can also be generalized to **open and twisted cases**, both for $U(1)$ -preserving and $U(1)$ -breaking

Outlook

- **The XYZ spin chain**

Q-system can be established
How to solve ?

- **Rigorous proof**

Prove the completeness of
Q-system rigourously

- **Increase efficiency**

Try to increase the efficiency of
the method as much as possible

