$D^{(2)}$ models and black holes

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Challenges in Integrability

Wuhan, China

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7 years ago, I was visiting Xiwen here; and one evening he told me: we should organize a meeting here.

You all know Xiwen: he works very hard, and he is very persuasive: it is very difficult to say “no” to him.

So, soon, many people got involved, and the plans for the meeting moved ahead well.

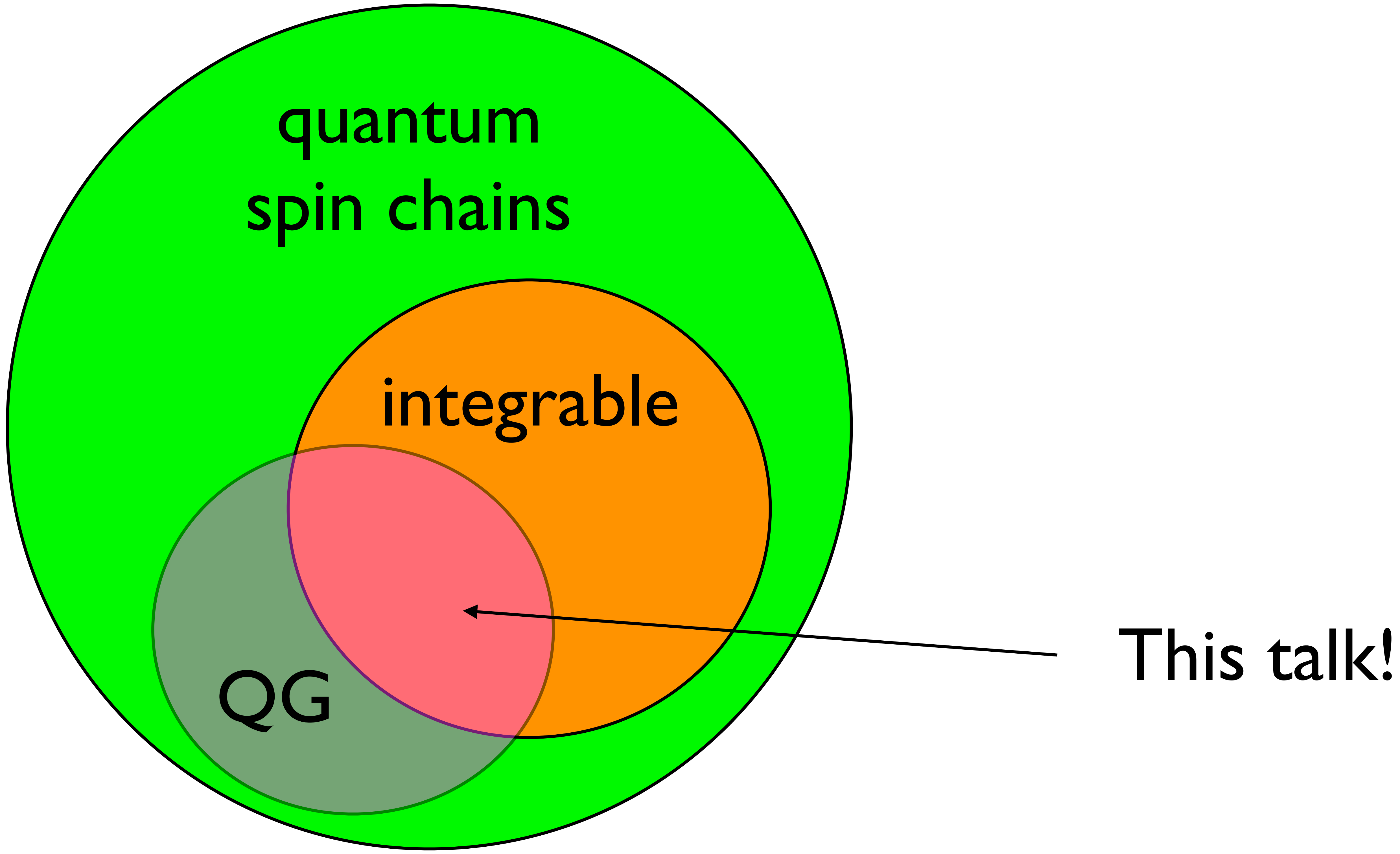
But then covid came; so the meeting was postponed for one year.

And then it was postponed again and again.

At some moment, I thought the meeting would never happen…

But, amazingly, thanks to Xiwen and the many other organizers, the meeting is finally happening!

THANK YOU!


quantum
spin chains
integrable
QG
This talk!

Outline

I. Review of quantum integrable models with quantum group symmetry

- Quantum integrability (sorry to the experts!)
- Symmetries & degeneracies of integrable spin chains (rank-1 algebra)

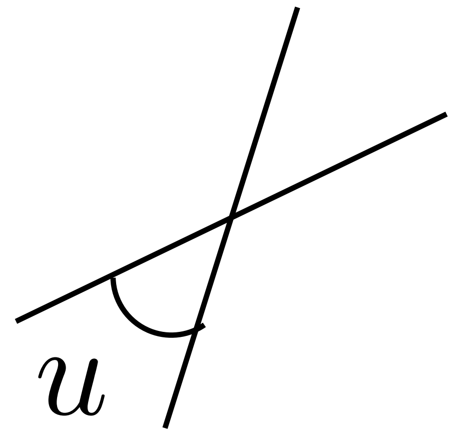
  $\longrightarrow$ *quantum group* symmetry

- Generalization to higher-rank algebras
- Bethe ansatz

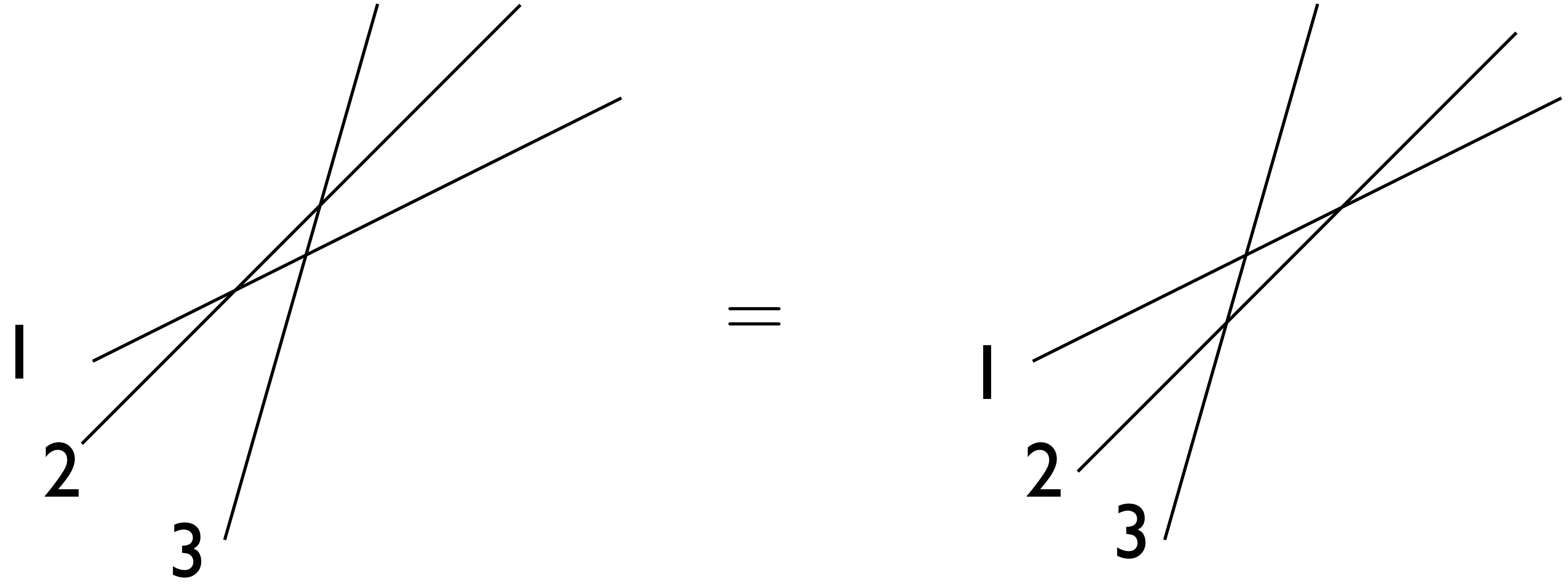
II. $D^{(2)}$ models

III. Outlook

Key to quantum integrability:

R-matrix $R(u) \sim$ 

Solution of Yang-Baxter equation (YBE)



$$R_{12}(u-v)\, R_{13}(u)\, R_{23}(v) = R_{23}(v)\, R_{13}(u)\, R_{12}(u-v)$$

Classes of R-matrices:

- Rational invariant under Lie group G

Ex: $$R(u)=\left(\begin{array}{cc|cc} u+i & 0 & 0 & 0 \\ 0 & u & i & 0 \\ \hline 0 & i & u & 0 \\ 0 & 0 & 0 & u+i \end{array}\right)$$

$G=SU(2)$

“isotropic”, XXX

- Trigonometric $\longleftrightarrow$ affine Lie algebra $\hat{g}$

Ex: $$R(u)=\left(\begin{array}{cc|cc} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh(u) & \sinh(\eta) & 0 \\ \hline 0 & \sinh(\eta) & \sinh(u) & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{array}\right)$$

$\hat{g}=A_1^{(1)}$

“anisotropic”, XXZ

η : “anisotropy parameter”

- Elliptic

Ex: $$R(u)=\left(\begin{array}{cc|cc} \mathrm{sn}(u+\eta) & 0 & 0 & k\,\mathrm{sn}(u+\eta)\,\mathrm{sn}(u)\,\mathrm{sn}(\eta) \\ 0 & \mathrm{sn}(u) & \mathrm{sn}(\eta) & 0 \\ \hline 0 & \mathrm{sn}(\eta) & \mathrm{sn}(u) & 0 \\ k\,\mathrm{sn}(u+\eta)\,\mathrm{sn}(u)\,\mathrm{sn}(\eta) & 0 & 0 & \mathrm{sn}(u+\eta) \end{array}\right)$$

XYZ

Uses of R-matrices:

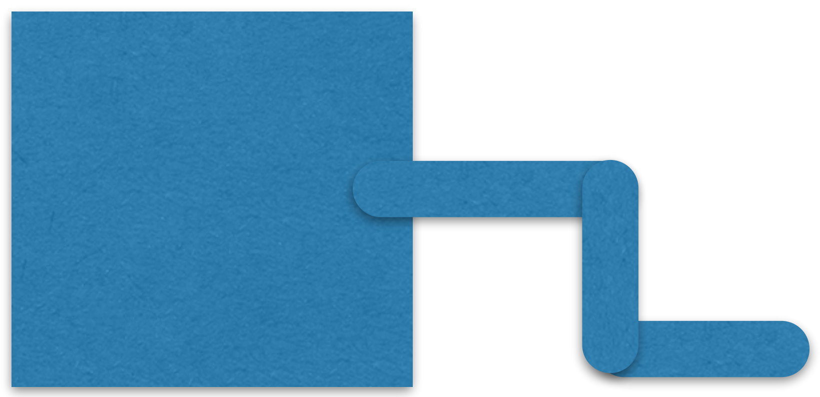
- S-matrices of integrable quantum field theories in 1+1 dimensions

  Ex: sine-Gordon model

- Building blocks of 1-dimensional integrable quantum spin chains

R-matrix

↓



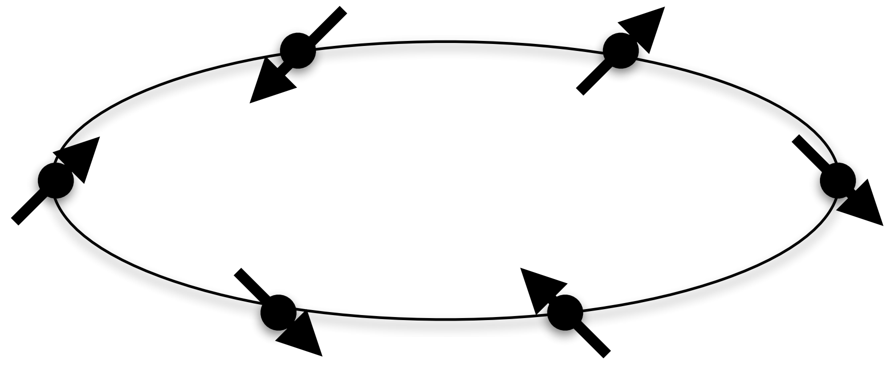
[Faddeev, … 1980's]

↓

integrable closed spin chain
Hamiltonian, transfer matrix

Ex: Heisenberg (XXX) spin chain

XXX R-matrix $\longrightarrow$ Hamiltonian:



N spin $\frac{1}{2}$

$$H^{(1)} = \sum_{n=1}^{N} \vec{\sigma}_n \cdot \vec{\sigma}_{n+1}\,, \quad \vec{\sigma}_{N+1} \equiv \vec{\sigma}_1$$

- quantum many-body model - many applications
- integrable:

$$H^{(2)} = \sum_{n=1}^{N} \vec{\sigma}_n \cdot \left(\vec{\sigma}_{n+1} \times \vec{\sigma}_{n+2}\right)$$

$$H^{(3)} = \ldots$$

$$\vdots$$

$$\boxed{\left[H^{(j)}\,, H^{(k)}\right] = 0}$$

$\rightsquigarrow$ solvable by Bethe ansatz

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- Rational [nice & straightforward]

R-matrix with symmetry G

$\Rightarrow$ The finite periodic spin chain has the same symmetry G

Ex: Heisenberg (XXX) spin chain has symmetry $SU(2)$

$$\boxed{\text{Degeneracies in the spectrum are given by group theory}}$$

Ex: N=4

Clebsch-Gordan $\Rightarrow$ $(\underline{\tfrac{1}{2}} \otimes \underline{\tfrac{1}{2}}) \otimes (\underline{\tfrac{1}{2}} \otimes \underline{\tfrac{1}{2}}) = (\underline{0} \oplus \underline{1}) \otimes (\underline{0} \oplus \underline{1})$

$$= 2 \cdot \underline{0} \oplus 3 \cdot \underline{1} \oplus \underline{2}$$

spin $s \longrightarrow 2s+1$

degeneracies $\{1, 1, 3, 3, 3, 5\}$ ✓

- Trigonometric [uninteresting?]

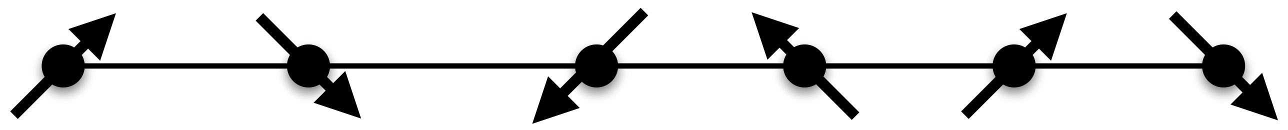
  $\hat{g}$ R-matrix

$\Rightarrow$

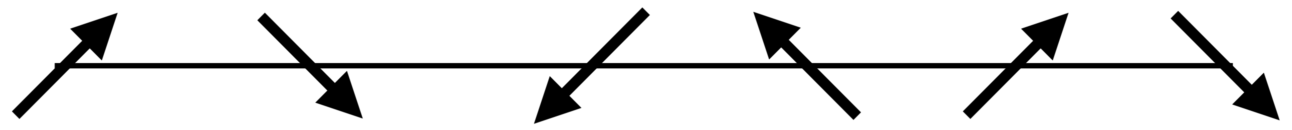
The finite periodic spin chain does *not* have any nice symmetry (beyond U(1)'s) or degeneracies

Periodic boundary conditions break symmetry

To realize symmetry, we consider *open* spin chain!



We insist that boundary conditions should preserve integrability

Example:  N spin $\frac{1}{2}$

$$H = \sum_{n=1}^{N-1} \left[\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \tfrac{1}{2}(q+q^{-1})\sigma_n^z \sigma_{n+1}^z\right] - \tfrac{1}{2}(q-q^{-1})\Big(\sigma_1^z - \sigma_N^z\Big)$$

[Pasquier, Saleur 1990]

- anisotropy parameter $q = e^{\eta}$
- integrable
- Quantum Group symmetry

Quantum Group (QG) symmetry

$$[H\,, S^z] = 0 = \left[H\,, S^\pm\right]$$

$$S^z = \sum_{k=1}^{N} S^z_k\,, \qquad S^z_k = \tfrac{1}{2}\sigma^z_k$$

"coproducts"

$$S^\pm = \sum_{k=1}^{N} q^{-(S^z_1+\cdots+S^z_{k-1})}\; S^\pm_k\; q^{(S^z_{k+1}+\cdots+S^z_N)}\,, \qquad S^\pm_k = \tfrac{1}{2}(\sigma^x_k \pm i\sigma^y_k)$$

$$[S^z, S^\pm] = \pm S^\pm, \qquad [S^+, S^-] = [2S^z]_q, \qquad [x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}}$$

$\rightarrow 2S^z$ for $q \rightarrow 1$

$$U_q sl(2) = U_q(A_1)$$

For generic q, the degeneracies are the same as for isotropic!

We would like to generalize this story to higher rank

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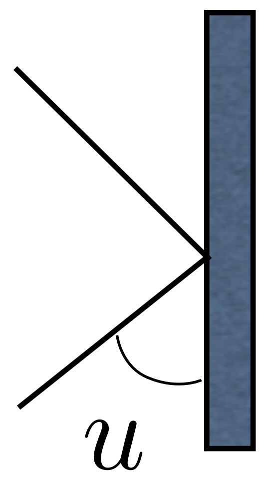
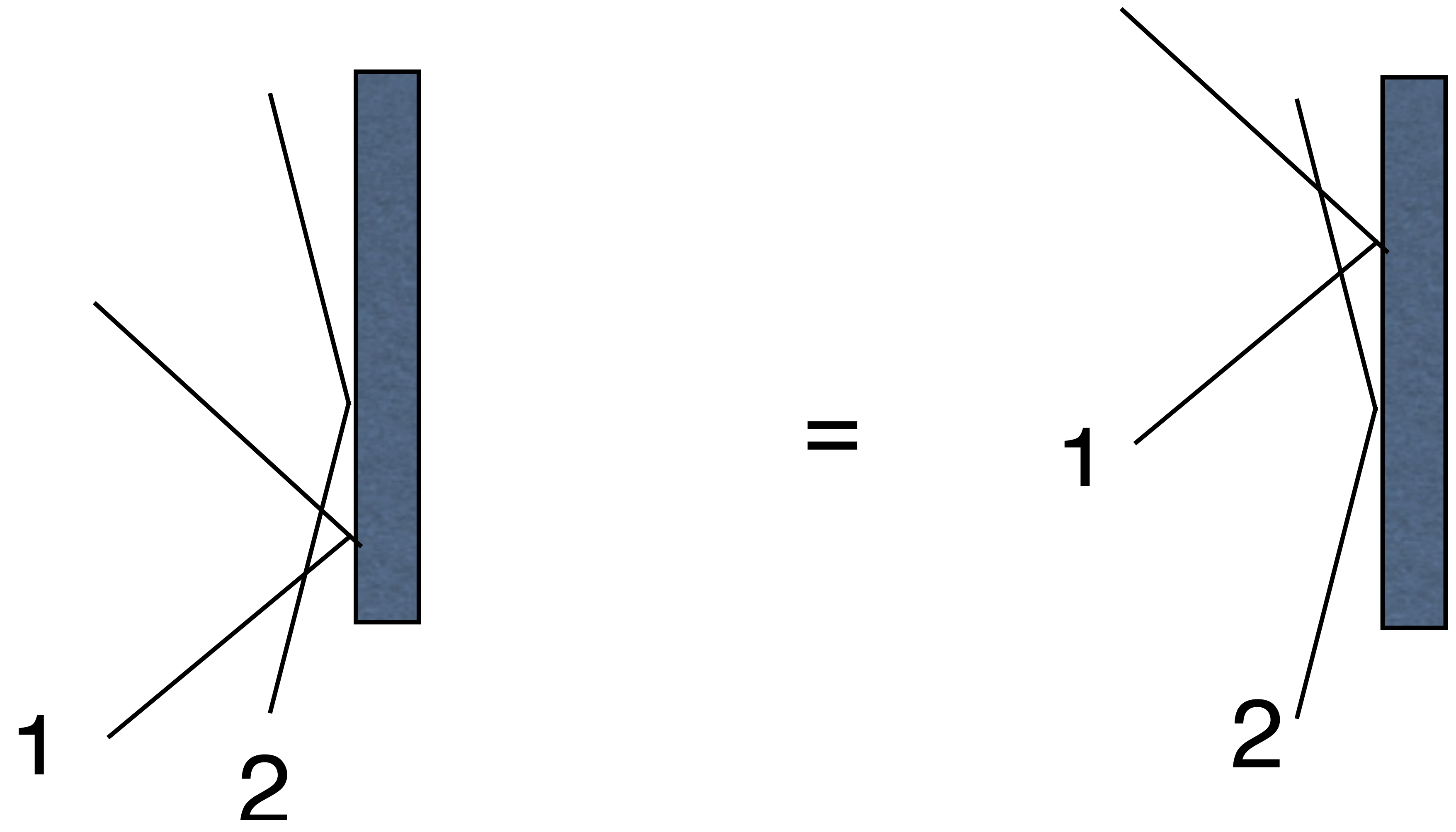
Key to *boundary* quantum integrability:

K-matrix (reflection matrix) $K^R(u) \sim$ 

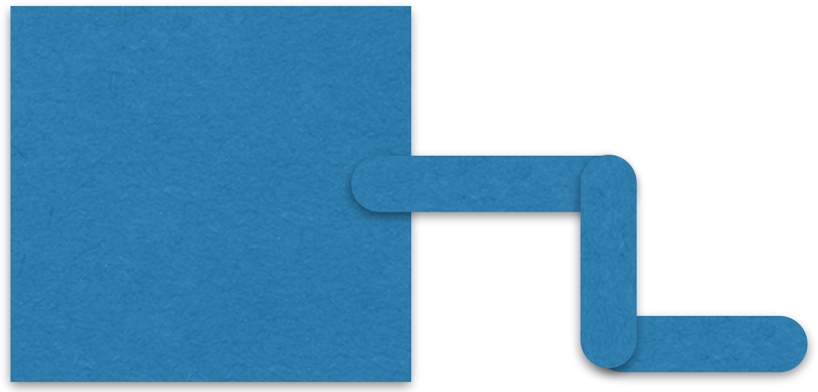
Solution of boundary Yang-Baxter equation (BYBE)



[Cherednik 1984]

$$R_{12}(u-v)\,K_1^R(u)\;R_{21}(u+v)\,K_2^R(v) = K_2^R(v)\,R_{12}(u+v)\,K_1^R(u)\,R_{21}(u-v)$$

R-matrix K-matrix



[Sklyanin 1988]

integrable *open* spin chain

Hamiltonian, transfer matrix

Let's consider integrable open spin chains constructed with:

- "trigonometric" R-matrices $\longleftrightarrow$ affine Lie algebras $\hat{g}$

$$\hat{g} = A_{2n}^{(2)}\,, A_{2n-1}^{(2)}\,, B_n^{(1)}\,, C_n^{(1)}\,, D_n^{(1)}\,, D_{n+1}^{(2)}$$

[Bazhanov 1985, Jimbo 1986]

- certain K-matrices, depending on discrete parameter $\boxed{\text{p}}$

[Batchelor et al 1996; Martins, Guan 2000; Lima-Santos 2006; RN, Pimenta 2018]

$$K^R(u,p) = \mathrm{diag}\,(\underbrace{e^{-u}, \ldots, e^{-u}}_{\boxed{p}}, *, \ldots, *, \underbrace{e^{u}, \ldots, e^{u}}_{\boxed{p}})$$

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2 x 2 block here for $\hat{g} = D_{n+1}^{(2)}$

Let's consider integrable open spin chains constructed with:

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$$\hat{g} = A^{(2)}_{2n}\,, A^{(2)}_{2n-1}\,, B^{(1)}_{n}\,, C^{(1)}_{n}\,, D^{(1)}_{n}\,, D^{(2)}_{n+1}$$ [Bazhanov 1985, Jimbo 1986]

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These spin chains have QG symmetry corresponding to removing pth node from $\hat{g}$ Dynkin diagram

[RN, Retore 2018]

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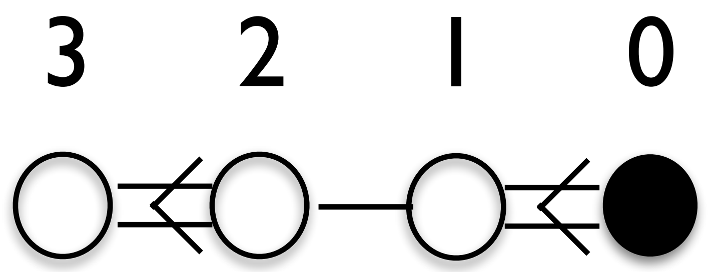
$$\boxed{\text{These spin chains have QG symmetry corresponding to removing p}^{\text{th}}\text{ node from } \hat{g} \text{ Dynkin diagram}}$$

[RN, Retore 2018]

$$p = 0 : \quad K^R(u, 0) = \mathbb{I}$$ [Mezincescu, RN 1991]

Ex: $\hat{g} = A_6^{(2)}$



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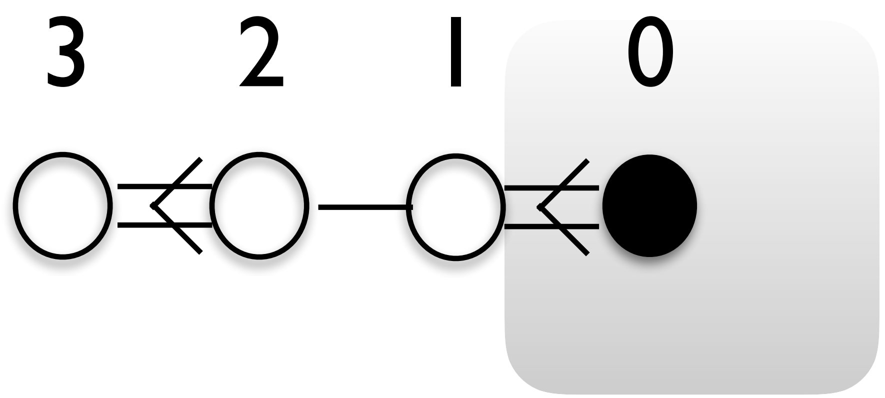
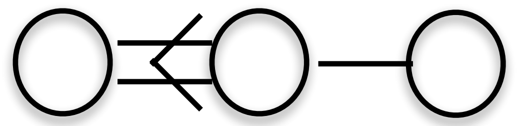


QG symmetry

$p = 0$

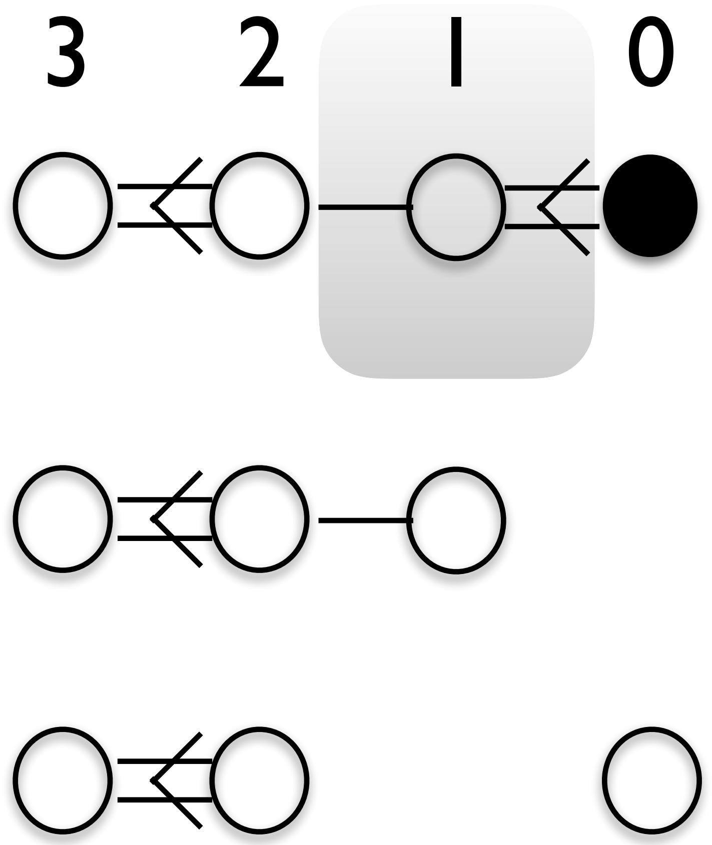


$U_q(B_3)$

Ex: $\hat{g} = A_6^{(2)}$

$p = 0$

$p = 1$

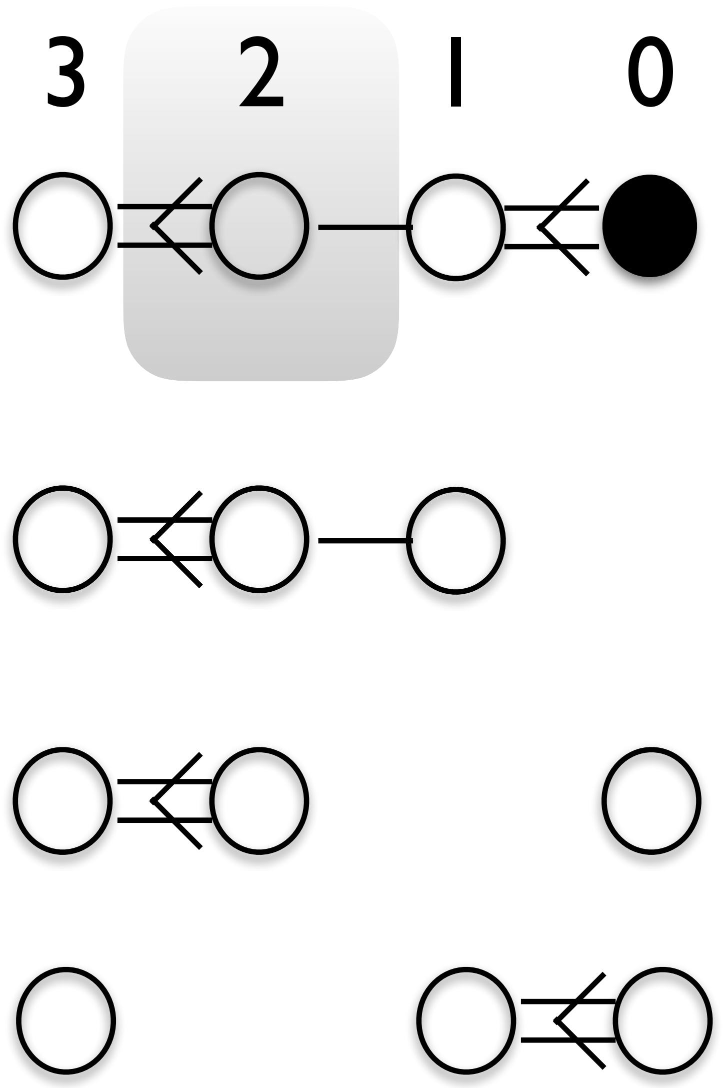


QG symmetry

$U_q(B_3)$

$U_q(B_2) \otimes U_q(C_1)$

Ex: $\hat{g} = A_6^{(2)}$



QG symmetry

$p = 0$ $\quad U_q(B_3)$

$p = 1$ $\quad U_q(B_2) \otimes U_q(C_1)$

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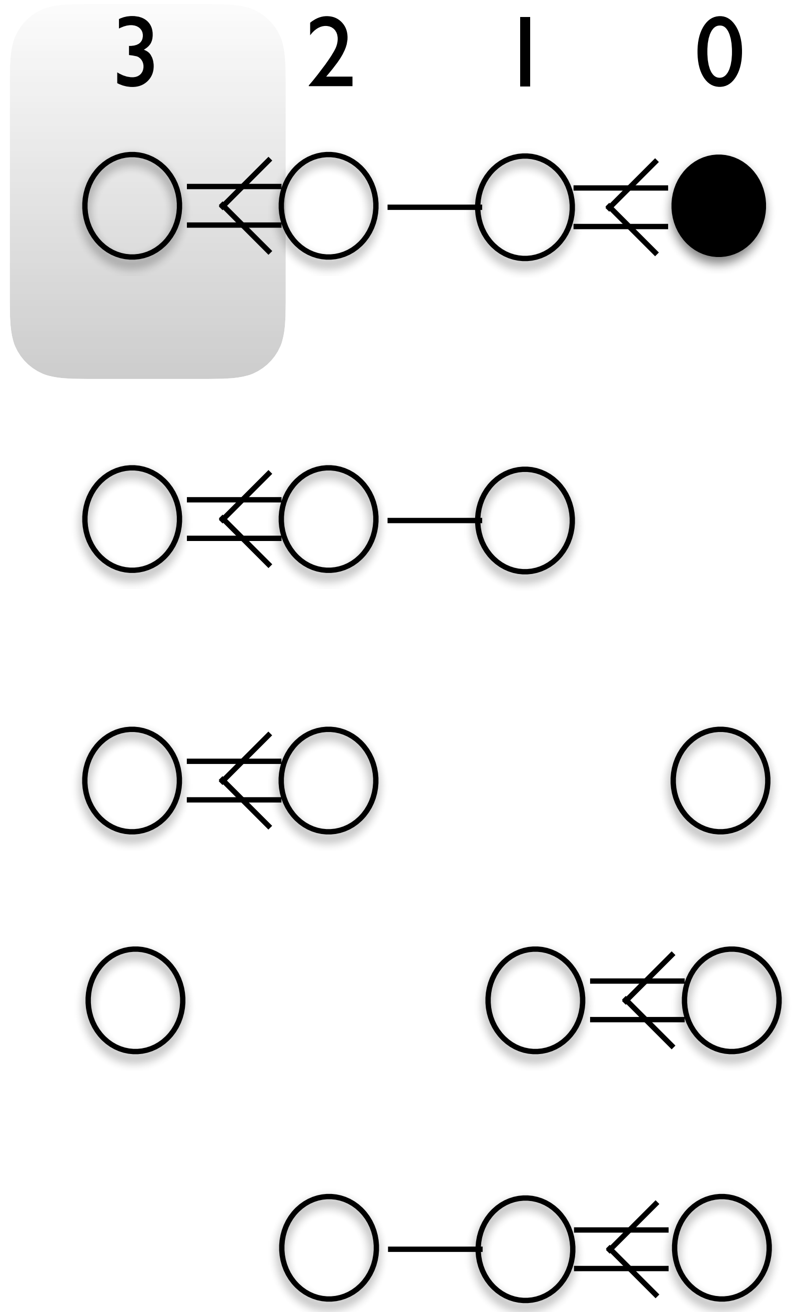

QG symmetry

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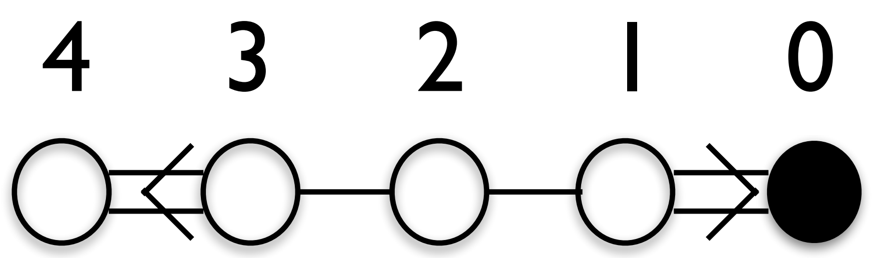
$p = 1$ $U_q(B_2) \otimes U_q(C_1)$

$p = 2$ $U_q(B_1) \otimes U_q(C_2)$

$p = 3$ $U_q(C_3)$

Ex: $\hat{g} = D_5^{(2)}$



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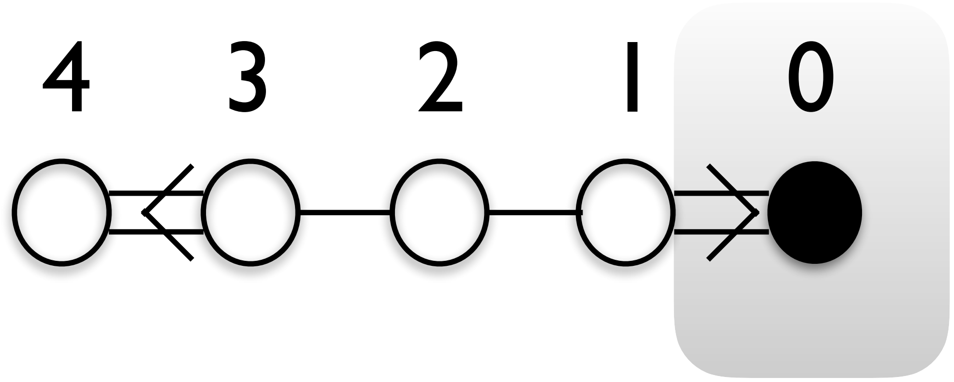

QG symmetry

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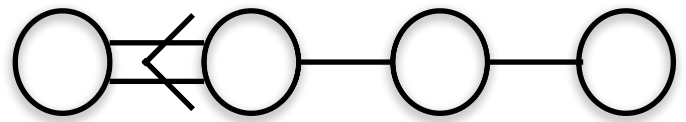
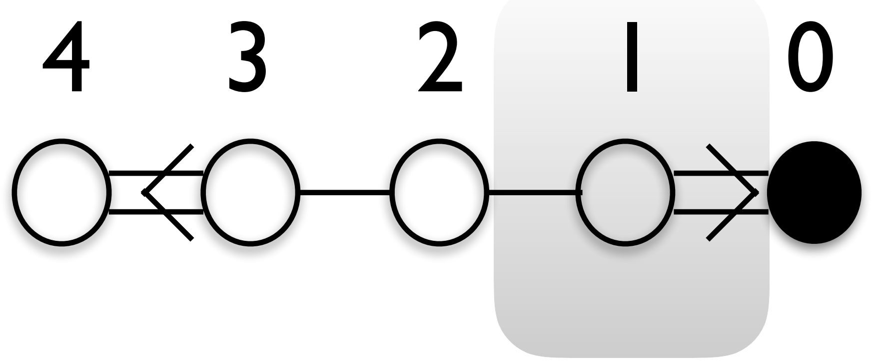


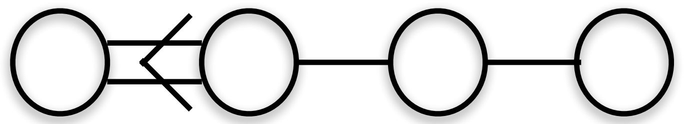
$U_q(B_4)$

Ex: $\hat{g} = D_5^{(2)}$



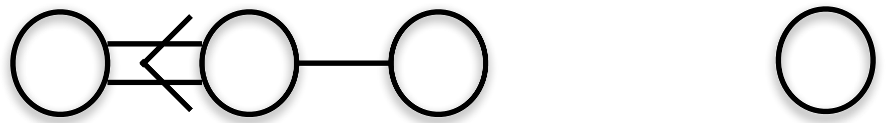
QG symmetry

$p = 0$



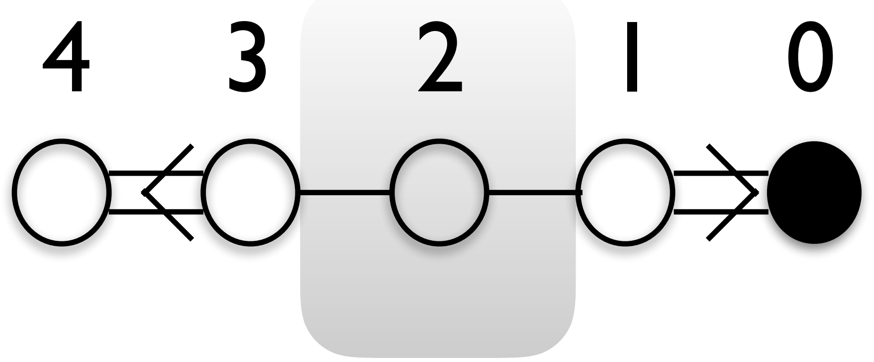
$U_q(B_4)$

$p = 1$



$U_q(B_3) \otimes U_q(B_1)$

Ex: $\hat{g} = D_5^{(2)}$



QG symmetry

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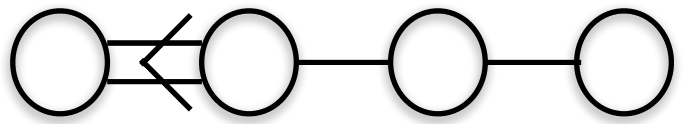

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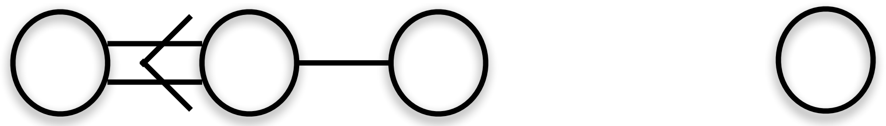
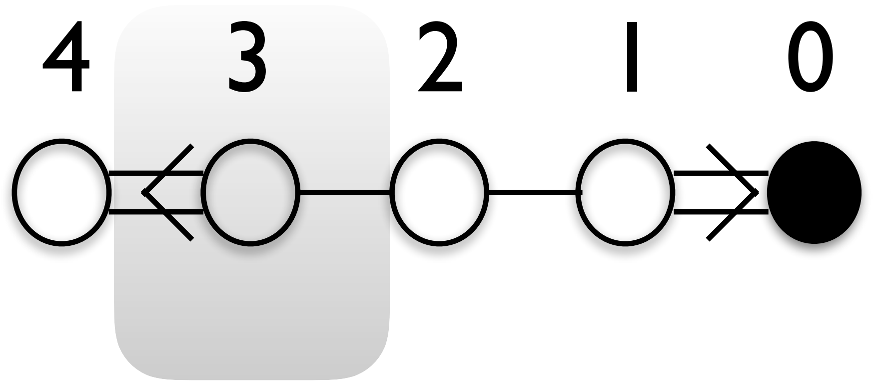


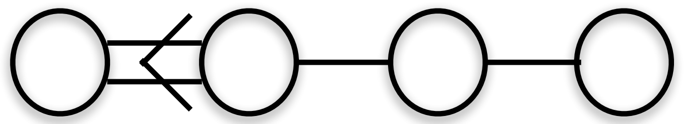
$U_q(B_3) \otimes U_q(B_1)$

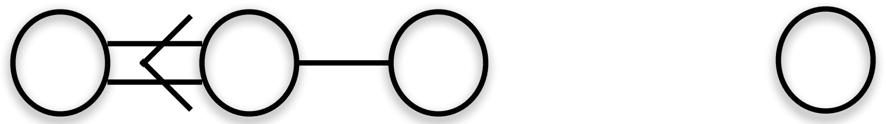
$p = 2$

$U_q(B_2) \otimes U_q(B_2)$

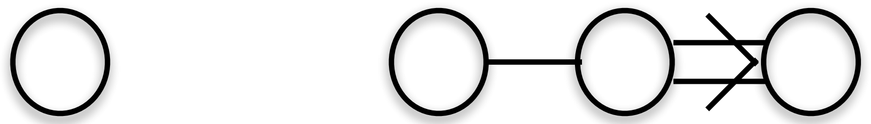
Ex: $\hat{g} = D_5^{(2)}$ 

QG symmetry

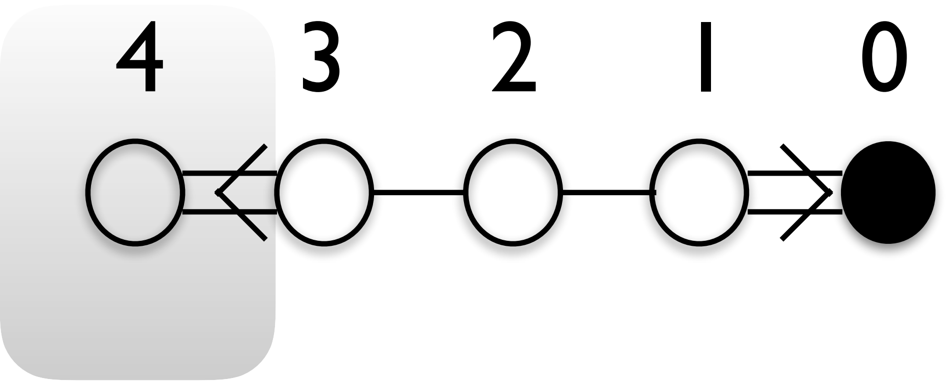
$p = 0$  $U_q(B_4)$

$p = 1$  $U_q(B_3) \otimes U_q(B_1)$

$p = 2$  $U_q(B_2) \otimes U_q(B_2)$

$p = 3$  $U_q(B_1) \otimes U_q(B_3)$

Ex: $\hat{g} = D_5^{(2)}$



QG symmetry

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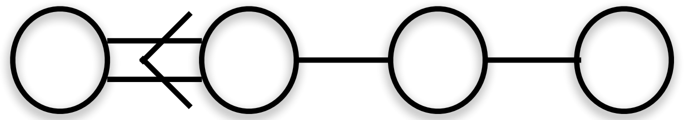
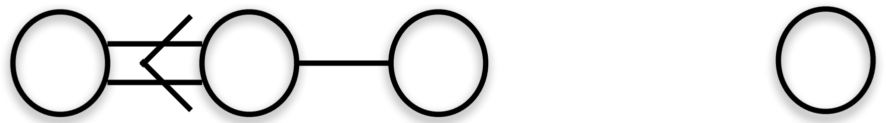


$U_q(B_4)$

$p = 1$

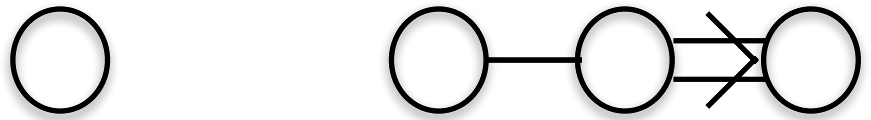


$U_q(B_3) \otimes U_q(B_1)$

$p = 2$

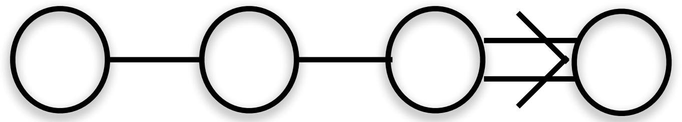
$U_q(B_2) \otimes U_q(B_2)$

$p = 3$



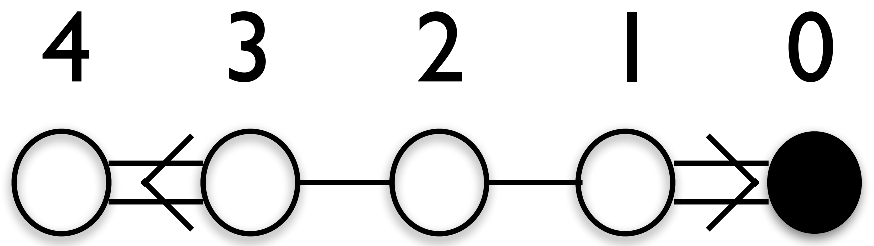
$U_q(B_1) \otimes U_q(B_3)$

$p = 4$

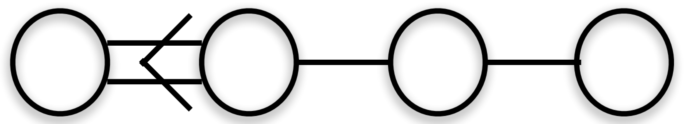


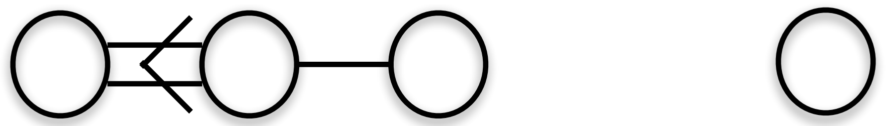
$U_q(B_4)$

Ex: $\hat{g} = D_5^{(2)}$



QG symmetry

$p = 0$  $U_q(B_4)$

$p = 1$  $U_q(B_3) \otimes U_q(B_1)$

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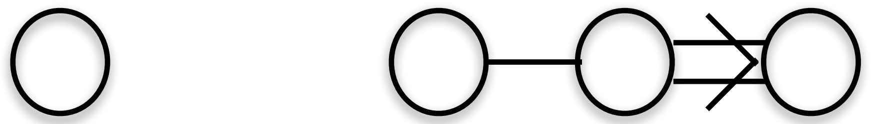
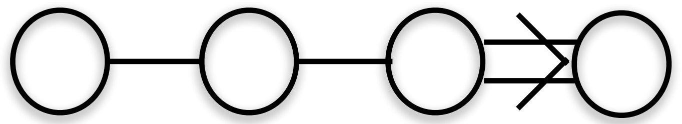
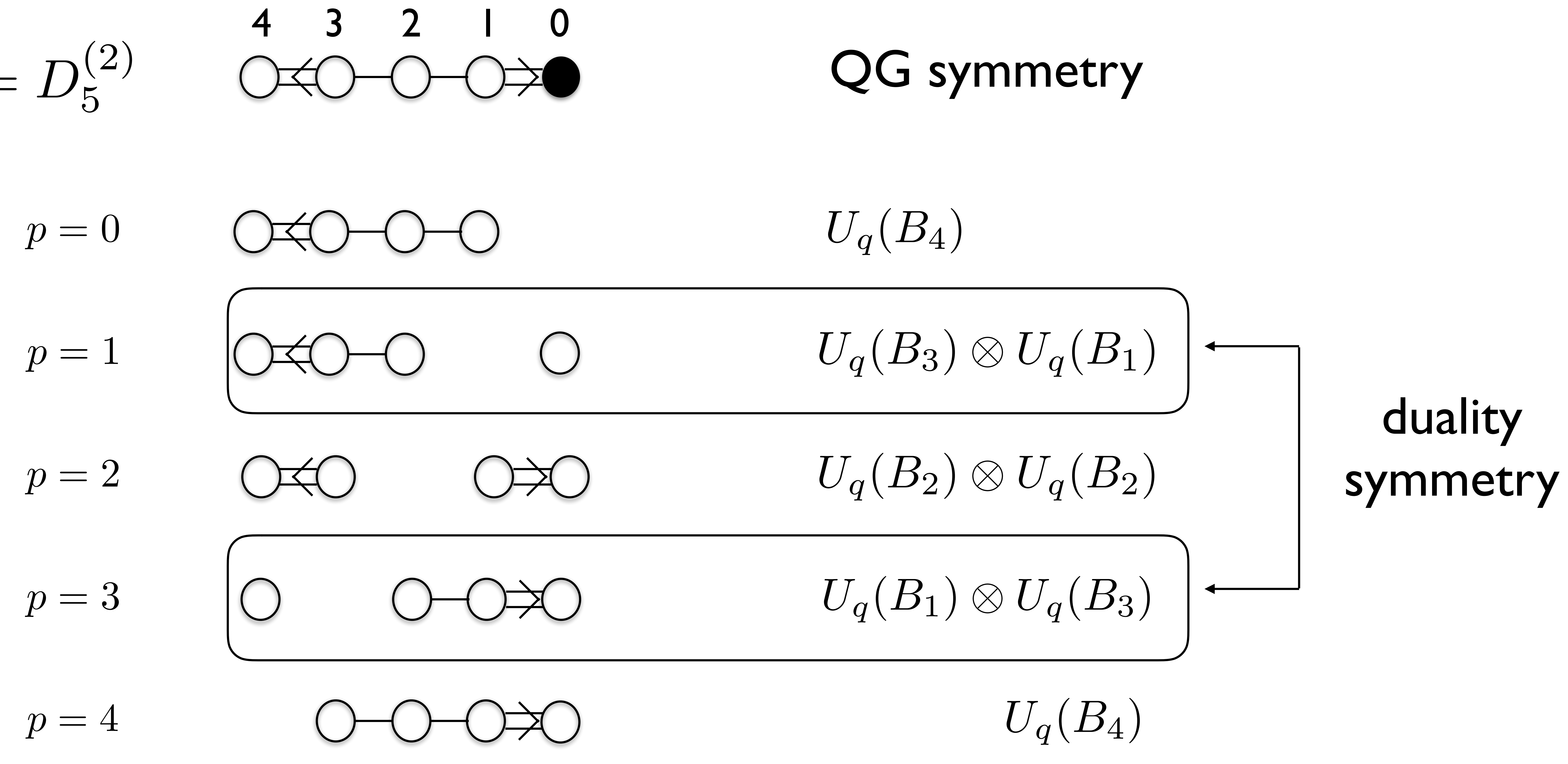
$p = 3$  $U_q(B_1) \otimes U_q(B_3)$

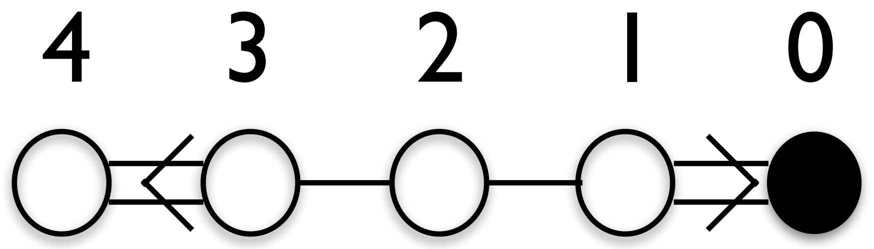
$p = 4$  $U_q(B_4)$

duality symmetry

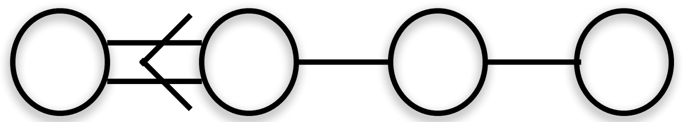
Ex: $\hat{g} = D_5^{(2)}$

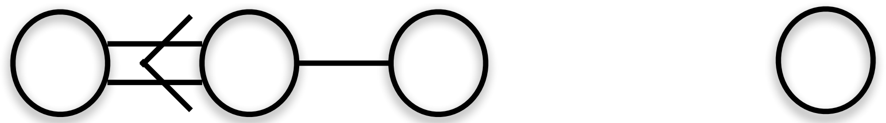

4 3 2 1 0
QG symmetry
$p = 0$
$U_q(B_4)$
$p = 1$
$U_q(B_3) \otimes U_q(B_1)$
$p = 2$
$U_q(B_2) \otimes U_q(B_2)$
$p = 3$
$U_q(B_1) \otimes U_q(B_3)$
$p = 4$
$U_q(B_4)$
duality
symmetry

Ex: $\hat{g} = D_5^{(2)}$

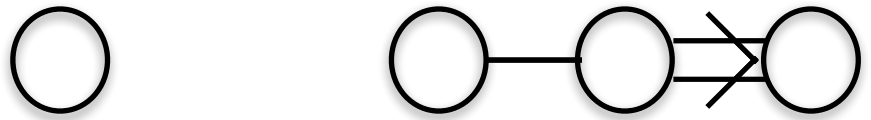


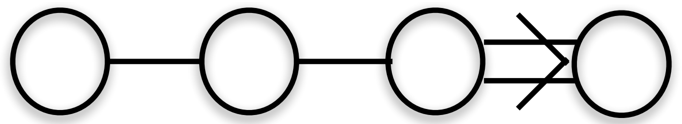
QG symmetry

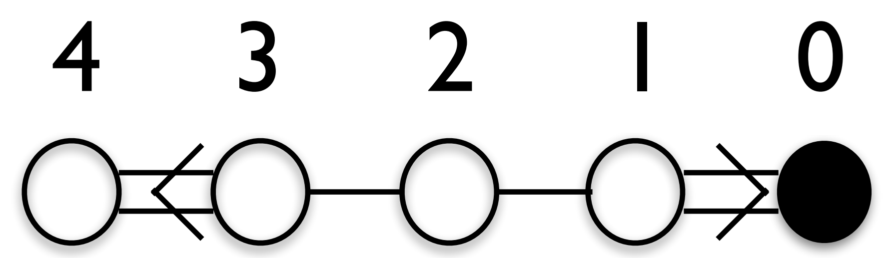
$p = 0$  $U_q(B_4)$

$p = 1$  $U_q(B_3) \otimes U_q(B_1)$

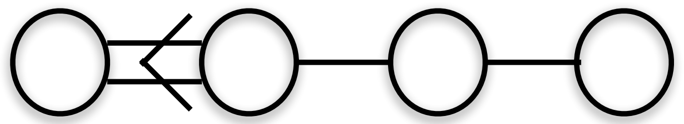
$p = 2$  $U_q(B_2) \otimes U_q(B_2)$ $\longleftrightarrow$ self-duality

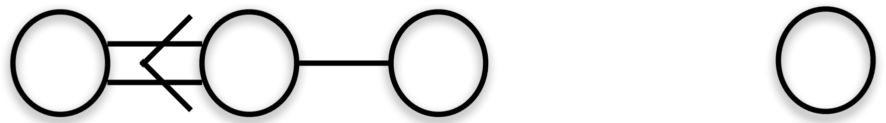
$p = 3$  $U_q(B_1) \otimes U_q(B_3)$

$p = 4$  $U_q(B_4)$

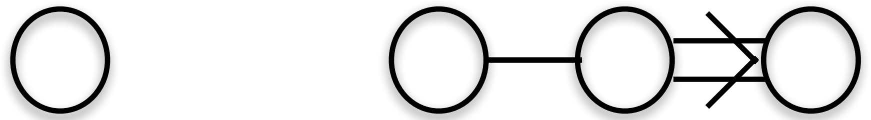
Ex: $\hat{g} = D_5^{(2)}$ 

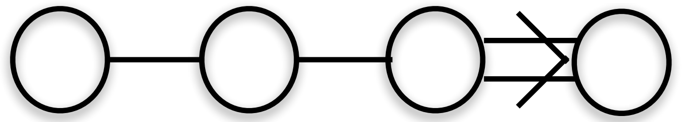
QG symmetry

$p = 0$  $U_q(B_4)$

$p = 1$  $U_q(B_3) \otimes U_q(B_1)$

$p = 2$  $U_q(B_2) \otimes U_q(B_2)$

$p = 3$  $U_q(B_1) \otimes U_q(B_3)$

$p = 4$  $U_q(B_4)$

$\hat{g} = D_{n+1}^{(2)}$ **duality symmetry** $p \leftrightarrow n - p$ $p = 0, \ldots, n$

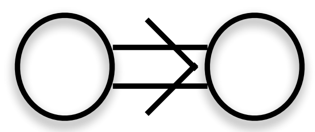
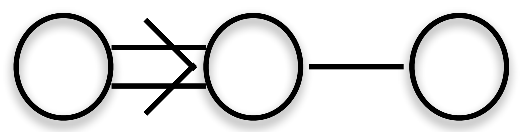
self-duality $p = \dfrac{n}{2}$ $n = \text{even}$

These symmetries explain the degeneracies of the transfer matrix!

Ex: $\hat{g} = A_{10}^{(2)}$ with $p = 3, N = 2$

By explicit diagonalization (generic q), find degeneracies:

$$\{1, 1, 10, 14, 14, 21, 30, 30\}$$

QG symmetry $U_q(B_2) \otimes U_q(C_3)$  

The "spin" at each site is in the representation $(\mathbf{5}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{6})$ 11-dimensional

LieART $\Rightarrow$

$$((\mathbf{5}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{6}))^{\otimes 2} = 2(\mathbf{1}, \mathbf{1}) \oplus 2(\mathbf{5}, \mathbf{6}) \oplus (\mathbf{10}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{14}) \oplus (\mathbf{14}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{21})$$

Exactly matches with observed degeneracies!

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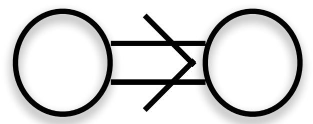
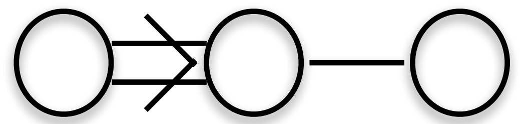
QG symmetry $U_q(B_2) \otimes U_q(C_3)$  

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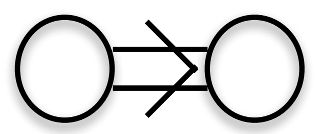
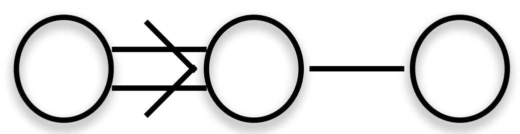
QG symmetry $U_q(B_2) \otimes U_q(C_3)$  

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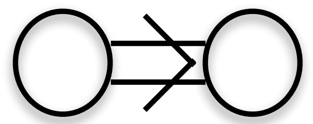
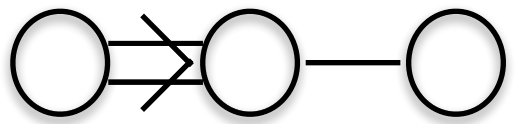
QG symmetry $U_q(B_2) \otimes U_q(C_3)$  

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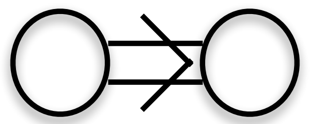
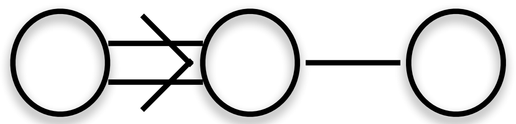
QG symmetry $U_q(B_2) \otimes U_q(C_3)$  

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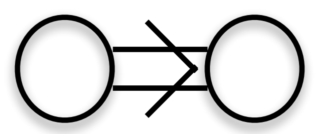
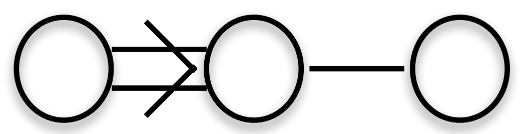
QG symmetry $U_q(B_2) \otimes U_q(C_3)$  

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Exactly matches with observed degeneracies! ✓

Outline

I. Review of quantum integrable models with quantum group symmetry

- Quantum integrability (sorry to the experts!)
- Symmetries & degeneracies of integrable spin chains (rank-1 algebra)

  $\longrightarrow$ *quantum group* symmetry

- Generalization to higher-rank algebras
- Bethe ansatz

II. $D^{(2)}$ models

III. Outlook

integrability $\Rightarrow$

can determine spectrum of transfer matrix by Bethe ansatz

Bethe equations: closed chain: [Reshetikhin1987]

$$\prod_{s=0}^{\boxed{t-1}} \left[\frac{\sinh\left(\frac{u_k^{(l)}}{2} + (\lambda_1, \theta^s \alpha_l)\,\eta + \frac{i\pi s}{2} \right)}{\sinh\left(\frac{u_k^{(l)}}{2} - (\lambda_1, \theta^s \alpha_l)\,\eta + \frac{i\pi s}{2} \right)} \right]^N$$

"universal"

$$= \prod_{s=0}^{\boxed{t-1}} \prod_{l'=1}^{n} \prod_{j=1}^{m_{l'}}{}' \frac{\sinh\left[\frac{1}{2}\left(u_k^{(l)} - u_j^{(l')} \right) + (\alpha_l, \theta^s \alpha_{l'})\,\eta + \frac{i\pi s}{2} \right]}{\sinh\left[\frac{1}{2}\left(u_k^{(l)} - u_j^{(l')} \right) - (\alpha_l, \theta^s \alpha_{l'})\,\eta + \frac{i\pi s}{2} \right]}$$

$$k = 1, \ldots, m_l\,, \qquad l = 1, \ldots, n$$

$t = 1$: untwisted

$t = 2$: twisted

integrability $\Rightarrow$

can determine spectrum of transfer matrix by Bethe ansatz

Bethe equations: open chain: [RN, Retore 2018]

$$\prod_{s=0}^{t-1}\left[\frac{\sinh\left(\frac{u_k^{(l)}}{2}+(\lambda_1,\theta^s\alpha_l)\,\eta+\frac{i\pi s}{2}\right)}{\sinh\left(\frac{u_k^{(l)}}{2}-(\lambda_1,\theta^s\alpha_l)\,\eta+\frac{i\pi s}{2}\right)}\right]^{2N}\Phi_{l,p,n}(u_k^{(l)})$$
$$=\prod_{s=0}^{t-1}\prod_{l'=1}^{n}\prod_{j=1}^{m_{l'}}{}' \frac{\sinh\left[\frac{1}{2}\left(u_k^{(l)}-u_j^{(l')}\right)+(\alpha_l,\theta^s\alpha_{l'})\,\eta+\frac{i\pi s}{2}\right]}{\sinh\left[\frac{1}{2}\left(u_k^{(l)}-u_j^{(l')}\right)-(\alpha_l,\theta^s\alpha_{l'})\,\eta+\frac{i\pi s}{2}\right]}\frac{\sinh\left[\frac{1}{2}\left(u_k^{(l)}+u_j^{(l')}\right)+(\alpha_l,\theta^s\alpha_{l'})\,\eta+\frac{i\pi s}{2}\right]}{\sinh\left[\frac{1}{2}\left(u_k^{(l)}+u_j^{(l')}\right)-(\alpha_l,\theta^s\alpha_{l'})\,\eta+\frac{i\pi s}{2}\right]}$$
$$k=1,\ldots,m_l\,,\qquad l=1,\ldots,n$$

$$(\hat{g}\neq D_{n+1}^{(2)})$$

$t=1$: untwisted

$t=2$: twisted

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$$= \prod_{s=0}^{t-1}\prod_{l'=1}^{n}\prod_{j=1}^{m_{l'}}{}' \frac{\sinh\left[\frac{1}{2}\left(u_k^{(l)}-u_j^{(l')}\right)+(\alpha_l,\theta^s\alpha_{l'})\,\eta+\frac{i\pi s}{2}\right]}{\sinh\left[\frac{1}{2}\left(u_k^{(l)}-u_j^{(l')}\right)-(\alpha_l,\theta^s\alpha_{l'})\,\eta+\frac{i\pi s}{2}\right]} \boxed{\frac{\sinh\left[\frac{1}{2}\left(u_k^{(l)}\boxed{+}u_j^{(l')}\right)+(\alpha_l,\theta^s\alpha_{l'})\,\eta+\frac{i\pi s}{2}\right]}{\sinh\left[\frac{1}{2}\left(u_k^{(l)}\boxed{+}u_j^{(l')}\right)-(\alpha_l,\theta^s\alpha_{l'})\,\eta+\frac{i\pi s}{2}\right]}}$$

$$k=1,\ldots,m_l\,,\qquad l=1,\ldots,n$$

"doubled" $\qquad (\hat{g}\neq D_{n+1}^{(2)})$

$t=1:$ untwisted

$t=2:$ twisted

integrability $\Rightarrow$

can determine spectrum of transfer matrix by Bethe ansatz

Bethe equations: open chain: [RN, Retore 2018]

$$\prod_{s=0}^{t-1}\left[\frac{\sinh\left(\frac{u_k^{(l)}}{2}+(\lambda_1,\theta^s\alpha_l)\,\eta+\frac{i\pi s}{2}\right)}{\sinh\left(\frac{u_k^{(l)}}{2}-(\lambda_1,\theta^s\alpha_l)\,\eta+\frac{i\pi s}{2}\right)}\right]^{2N} \boxed{\Phi_{l,p,n}(u_k^{(l)})} \longleftarrow \neq 1 \text{ only if } l=p$$

$$=\prod_{s=0}^{t-1}\prod_{l'=1}^{n}\prod_{j=1}^{m_{l'}}{}' \frac{\sinh\left[\frac{1}{2}\left(u_k^{(l)}-u_j^{(l')}\right)+(\alpha_l,\theta^s\alpha_{l'})\,\eta+\frac{i\pi s}{2}\right]}{\sinh\left[\frac{1}{2}\left(u_k^{(l)}-u_j^{(l')}\right)-(\alpha_l,\theta^s\alpha_{l'})\,\eta+\frac{i\pi s}{2}\right]}\frac{\sinh\left[\frac{1}{2}\left(u_k^{(l)}+u_j^{(l')}\right)+(\alpha_l,\theta^s\alpha_{l'})\,\eta+\frac{i\pi s}{2}\right]}{\sinh\left[\frac{1}{2}\left(u_k^{(l)}+u_j^{(l')}\right)-(\alpha_l,\theta^s\alpha_{l'})\,\eta+\frac{i\pi s}{2}\right]}$$

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$$k=1,\ldots,m_l\,,\qquad l=1,\ldots,n$$

$$\boxed{(\hat{g}\neq D_{n+1}^{(2)})}$$

$t=1:$ untwisted

$t=2:$ twisted

So, how about $D_{n+1}^{(2)}$?

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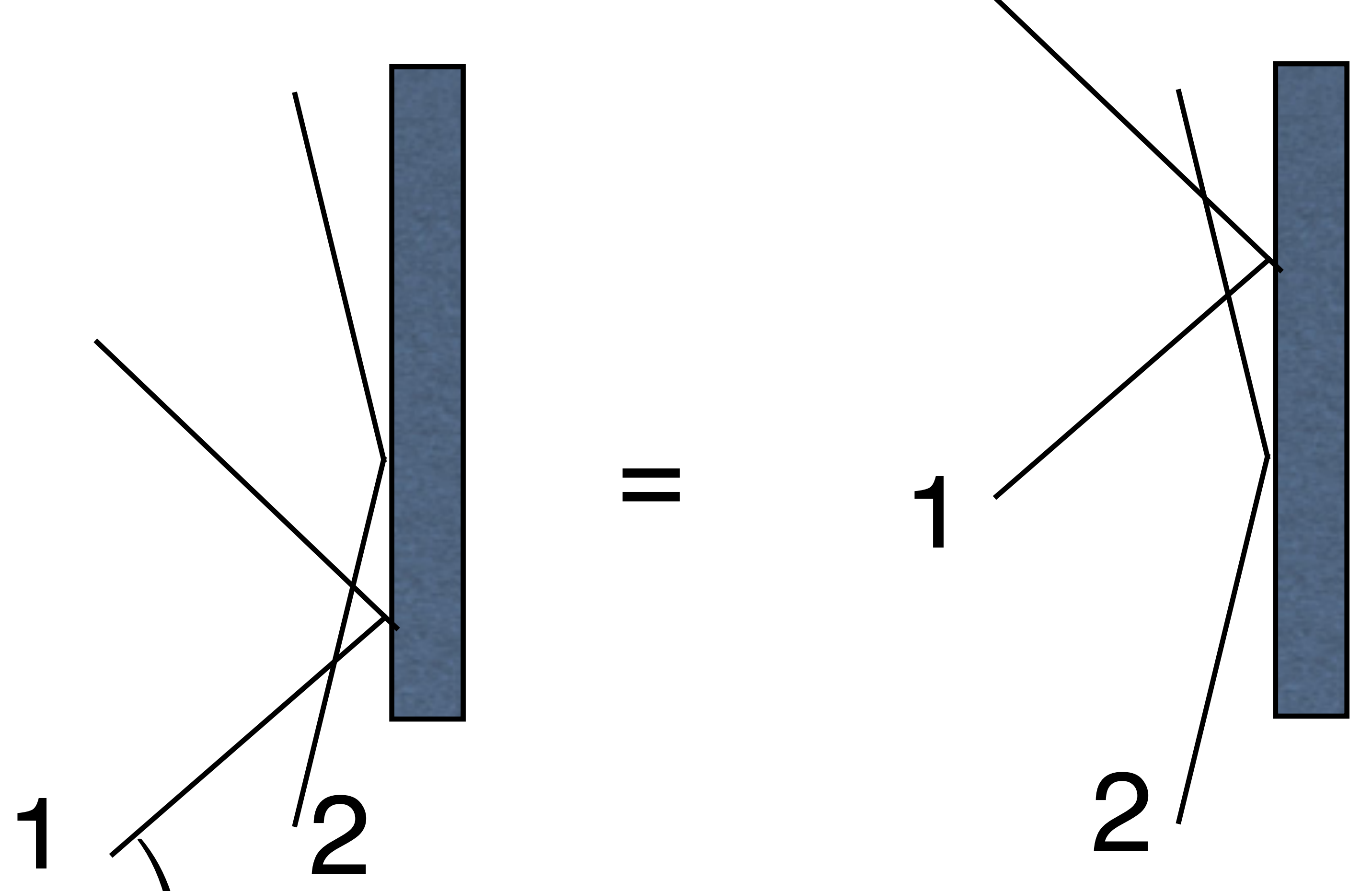
Boundary Yang-Baxter equation

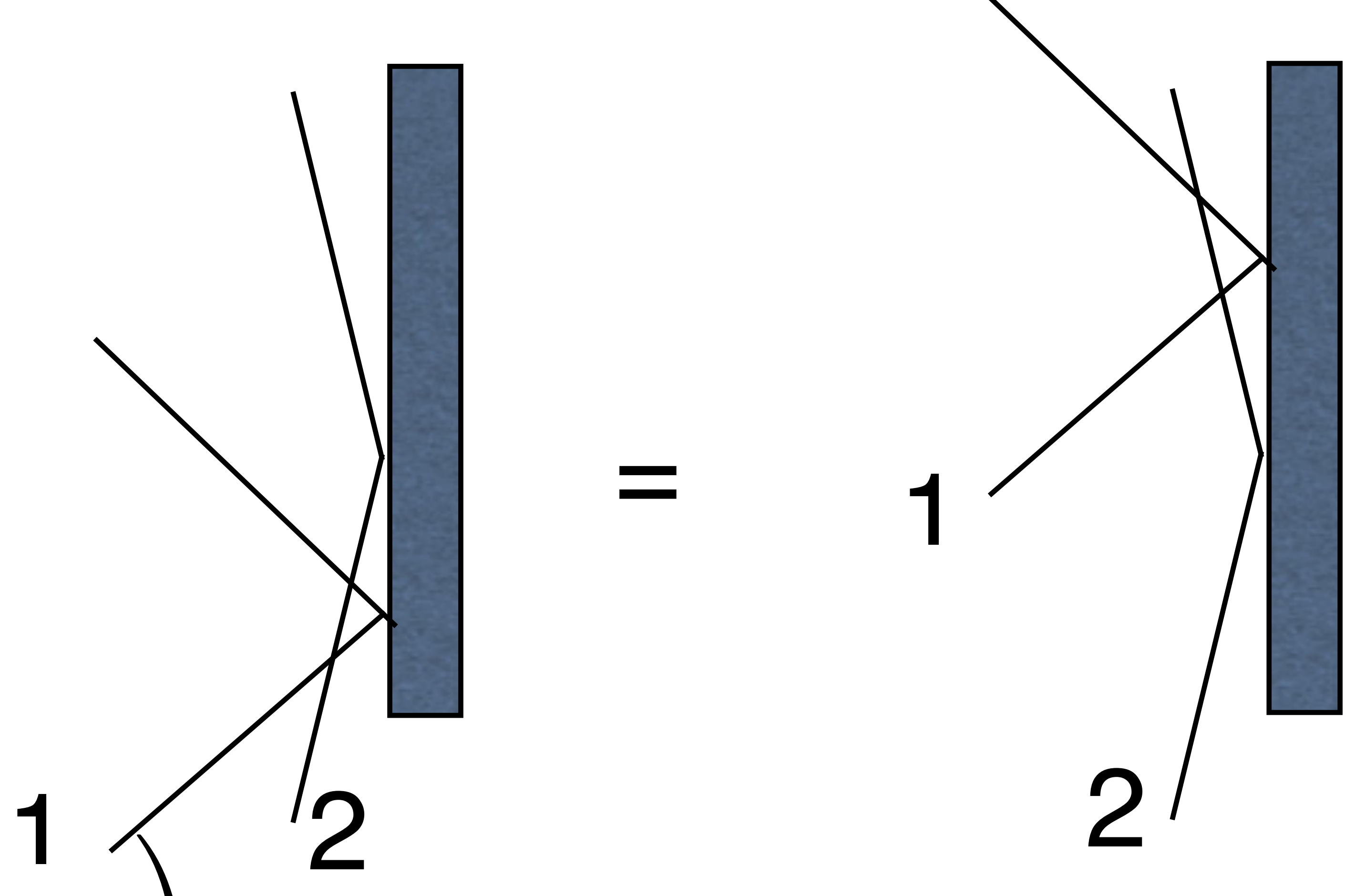


$$\hat{g} = D^{(2)}_{n+1}:$$

$$K^R(u) = \left(\begin{array}{c|c|c|c|c}
k_-(u)\,\mathbb{I}_{p\times p} & & & & \\
\hline
 & g(u)\,\mathbb{I}_{(n-p)\times(n-p)} & & & \\
\hline
 & & \begin{matrix} k_1(u) & k_2(u) \\ k_2(u) & k_1(u) \end{matrix} & & \\
\hline
 & & & g(u)\,\mathbb{I}_{(n-p)\times(n-p)} & \\
\hline
 & & & & k_+(u)\,\mathbb{I}_{p\times p}
\end{array}\right)$$

[Martins, Guan 2000;
RN, Pimenta 2018]

Boundary Yang-Baxter equation



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[Martins, Guan 2000;
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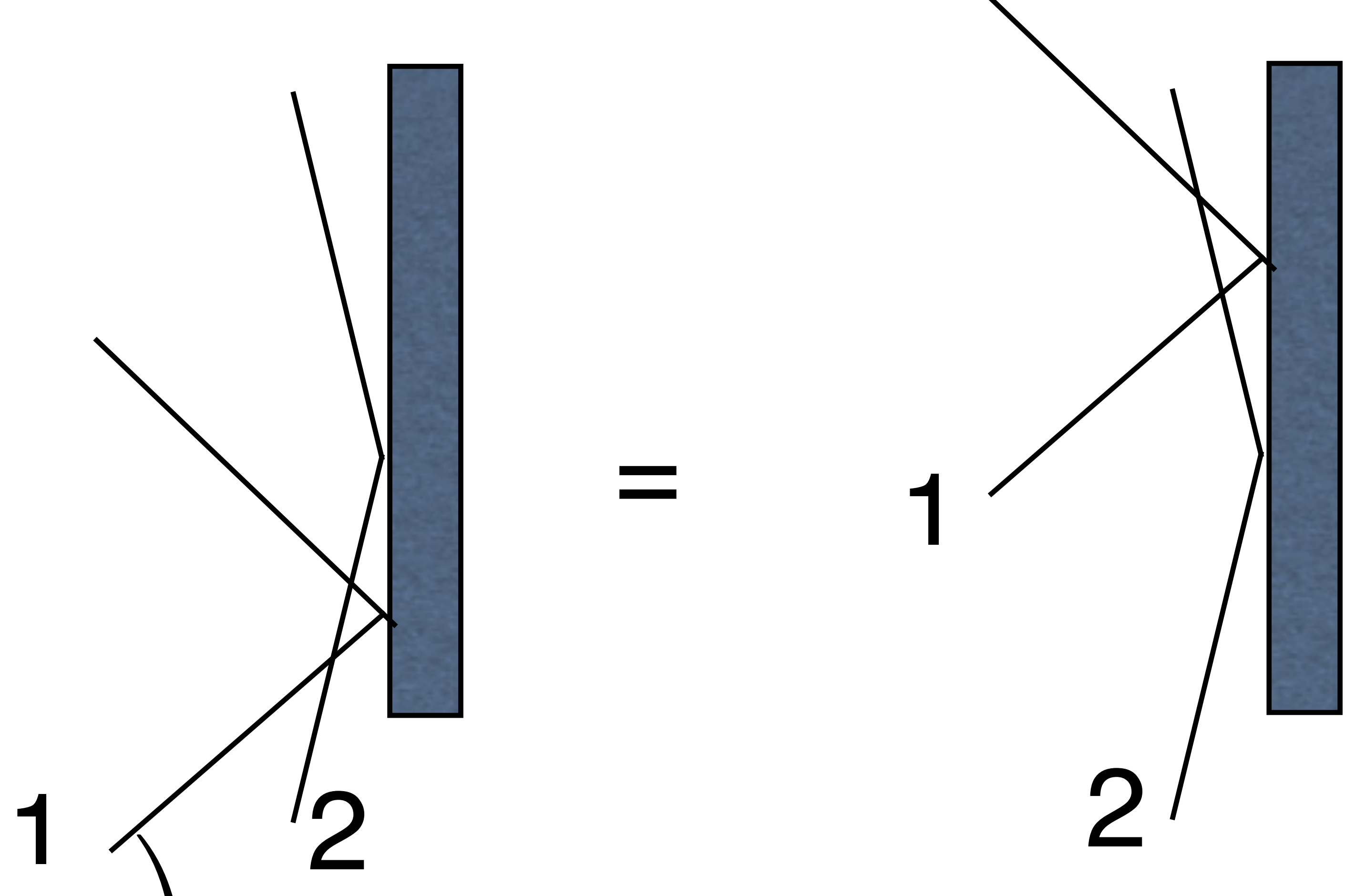
$$p = 0, \ldots, n$$

Boundary Yang-Baxter equation



$\hat{g} = D^{(2)}_{\boxed{n}+1}$:

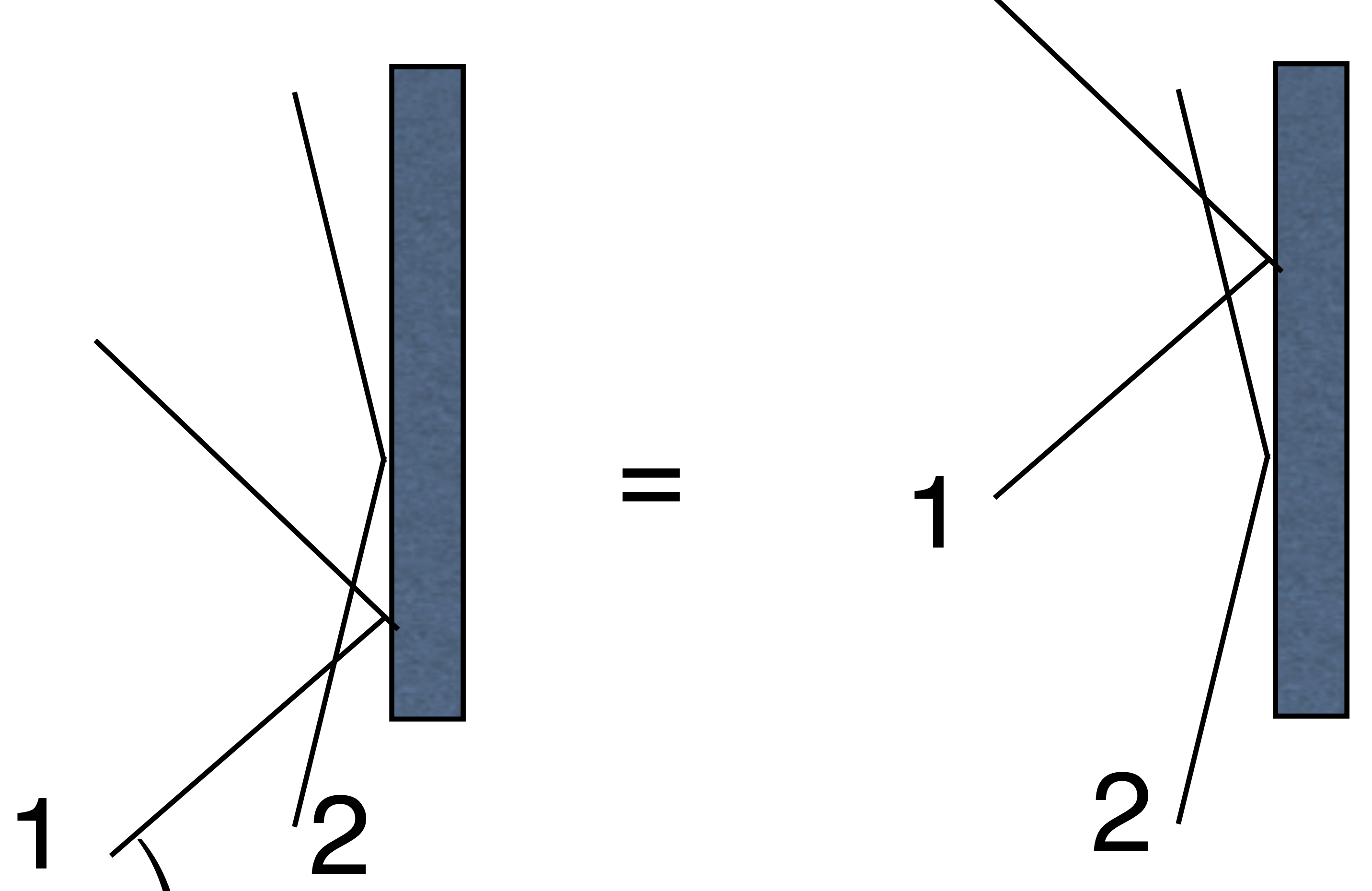
$$K^R(u) = \left(\begin{array}{c|c|c|c|c} k_-(u)\,\mathbb{I}_{p\times p} & & & & \\ \hline & g(u)\,\mathbb{I}_{(n-p)\times(n-p)} & & & \\ \hline & & \begin{array}{cc} k_1(u) & k_2(u) \\ k_2(u) & k_1(u) \end{array} & & \\ \hline & & & g(u)\,\mathbb{I}_{(n-p)\times(n-p)} & \\ \hline & & & & k_+(u)\,\mathbb{I}_{p\times p} \end{array}\right) \quad (2n+2)\times(2n+2)$$

[Martins, Guan 2000;
RN, Pimenta 2018]

$$p = 0, \ldots, n$$

$$n = 1, 2, \ldots$$

Boundary Yang-Baxter equation



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 & & k_1(u) & k_2(u) & & \\
 & & k_2(u) & k_1(u) & & \\ \hline
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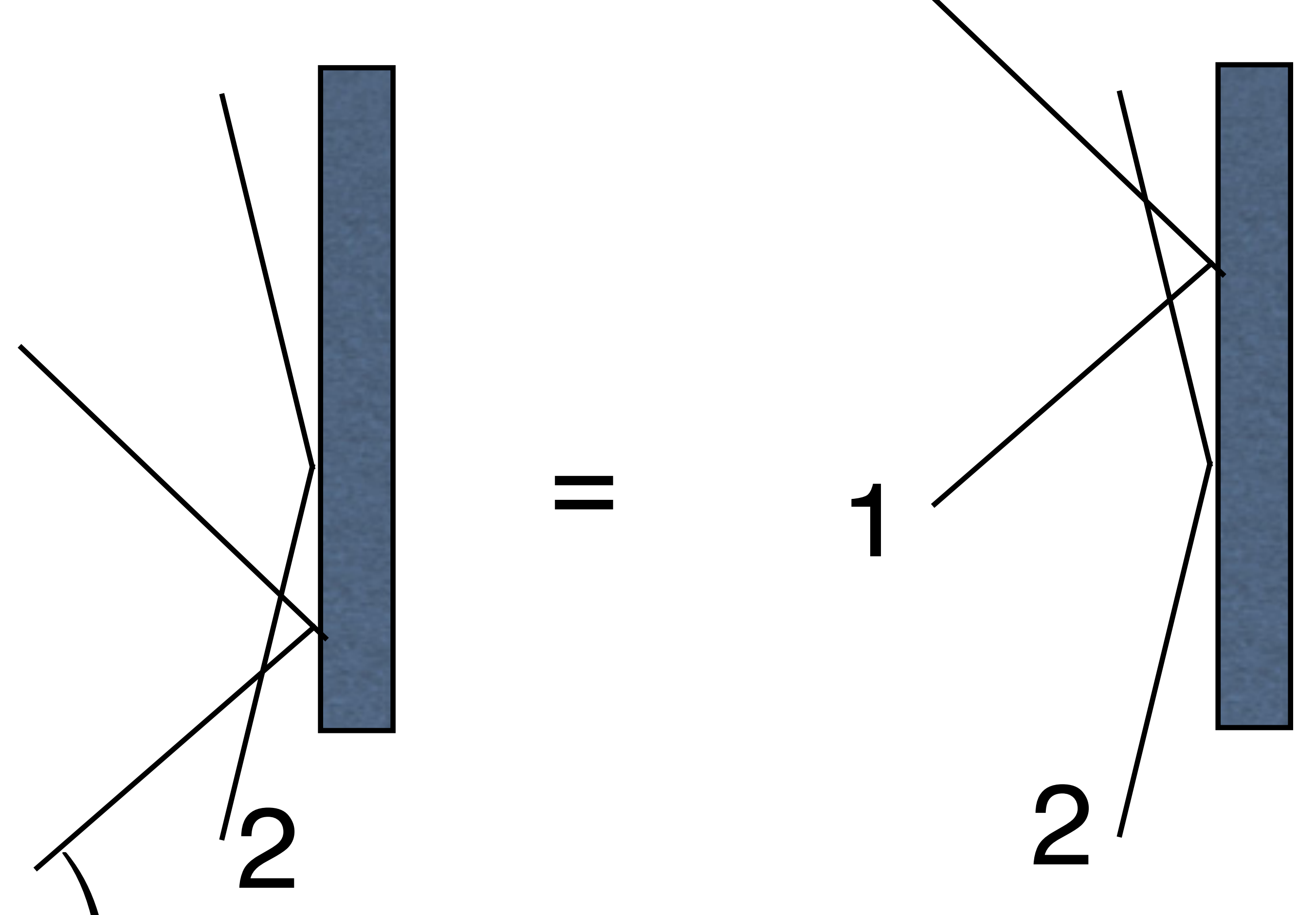
 & & & & & k_+(u)\,\mathbb{I}_{p\times p}
\end{array}\right)$$

[Martins, Guan 2000;
RN, Pimenta 2018]

QG symmetry $U_q(B_{n-p}) \otimes U_q(B_p)$ $\qquad p = 0, \ldots, n$

duality symmetry $p \leftrightarrow n - p$ $\qquad n = 1, 2, \ldots$

Boundary Yang-Baxter equation



$\hat{g} = D^{(2)}_{n+1}$ **:**

$$K^R(u) = \left(\begin{array}{c|c|c|c|c}
\boxed{k_-(u)}\mathbb{I}_{p\times p} & & & & \\
\hline
 & \boxed{g(u)}\mathbb{I}_{(n-p)\times(n-p)} & & & \\
\hline
 & & \boxed{\begin{matrix} k_1(u) & k_2(u) \\ k_2(u) & k_1(u) \end{matrix}} & & \\
\hline
 & & & \boxed{g(u)}\mathbb{I}_{(n-p)\times(n-p)} & \\
\hline
 & & & & \boxed{k_+(u)}\mathbb{I}_{p\times p}
\end{array}\right)$$

[Martins, Guan 2000; RN, Pimenta 2018]

$$k_\mp(u) = e^{\mp 2u}$$

$$g(u) = \frac{\cosh(u - (n-2p)\eta + \frac{i\pi}{2}\varepsilon)}{\cosh(u + (n-2p)\eta - \frac{i\pi}{2}\varepsilon)}$$

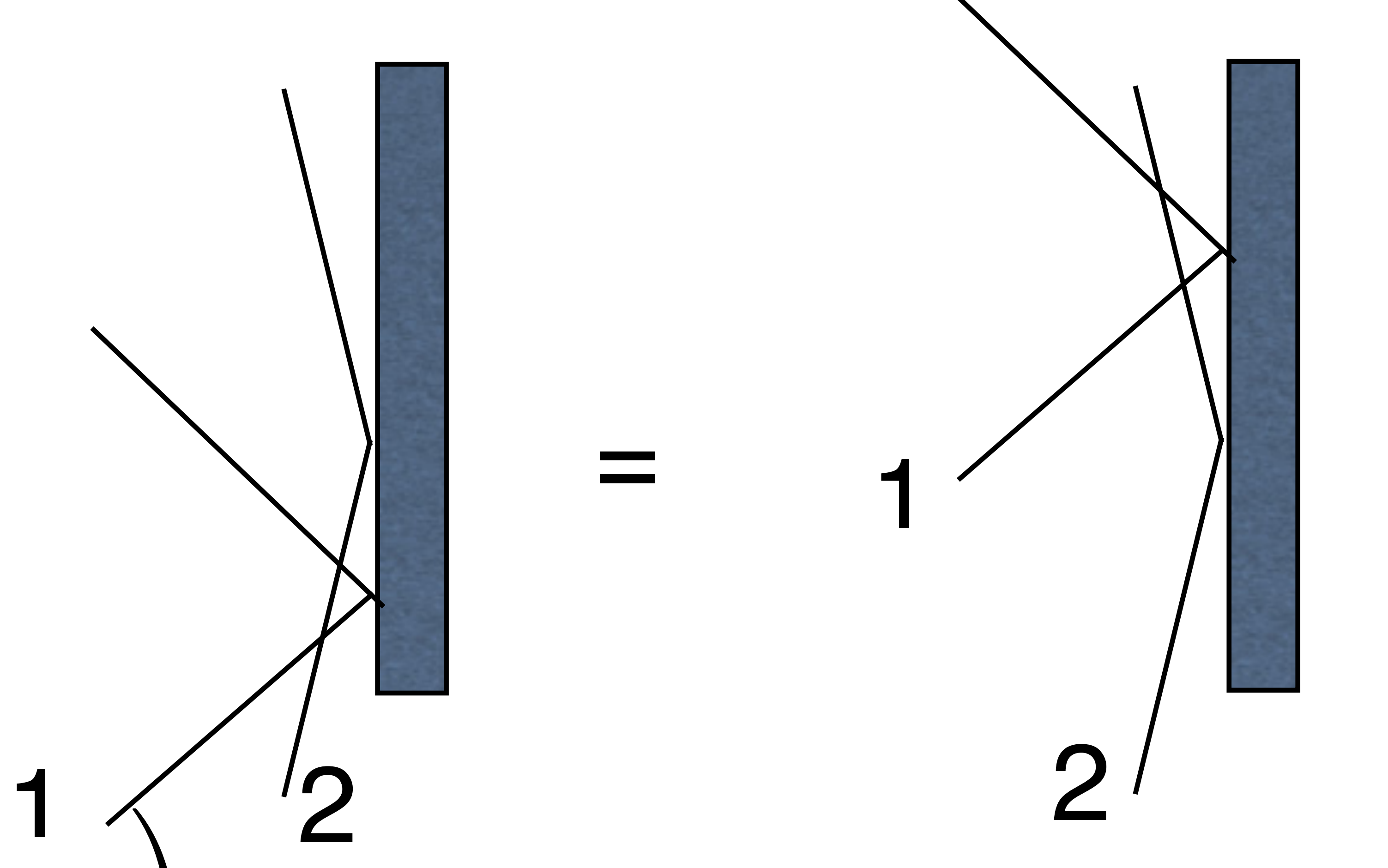
$$k_1(u) = \frac{\cosh(u)\cosh((n-2p)\eta + \frac{i\pi}{2}\varepsilon)}{\cosh(u + (n-2p)\eta) + \frac{i\pi}{2}\varepsilon)}$$

$$k_2(u) = -\frac{\sinh(u)\sinh((n-2p)\eta + \frac{i\pi}{2}\varepsilon)}{\cosh(u + (n-2p)\eta + \frac{i\pi}{2}\varepsilon)}$$

$$p = 0, \ldots, n$$

$$n = 1, 2, \ldots$$

Boundary Yang-Baxter equation



$\hat{g} = D^{(2)}_{n+1}$:

$$
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 & & k_1(u) & k_2(u) & & \\
 & & k_2(u) & k_1(u) & & \\ \hline
 & & & & g(u)\,\mathbb{I}_{(n-p)\times(n-p)} & \\ \hline
 & & & & & k_+(u)\,\mathbb{I}_{p\times p}
\end{array}\right)
$$

[Martins, Guan 2000; RN, Pimenta 2018]

$$k_\mp(u) = e^{\mp 2u}$$

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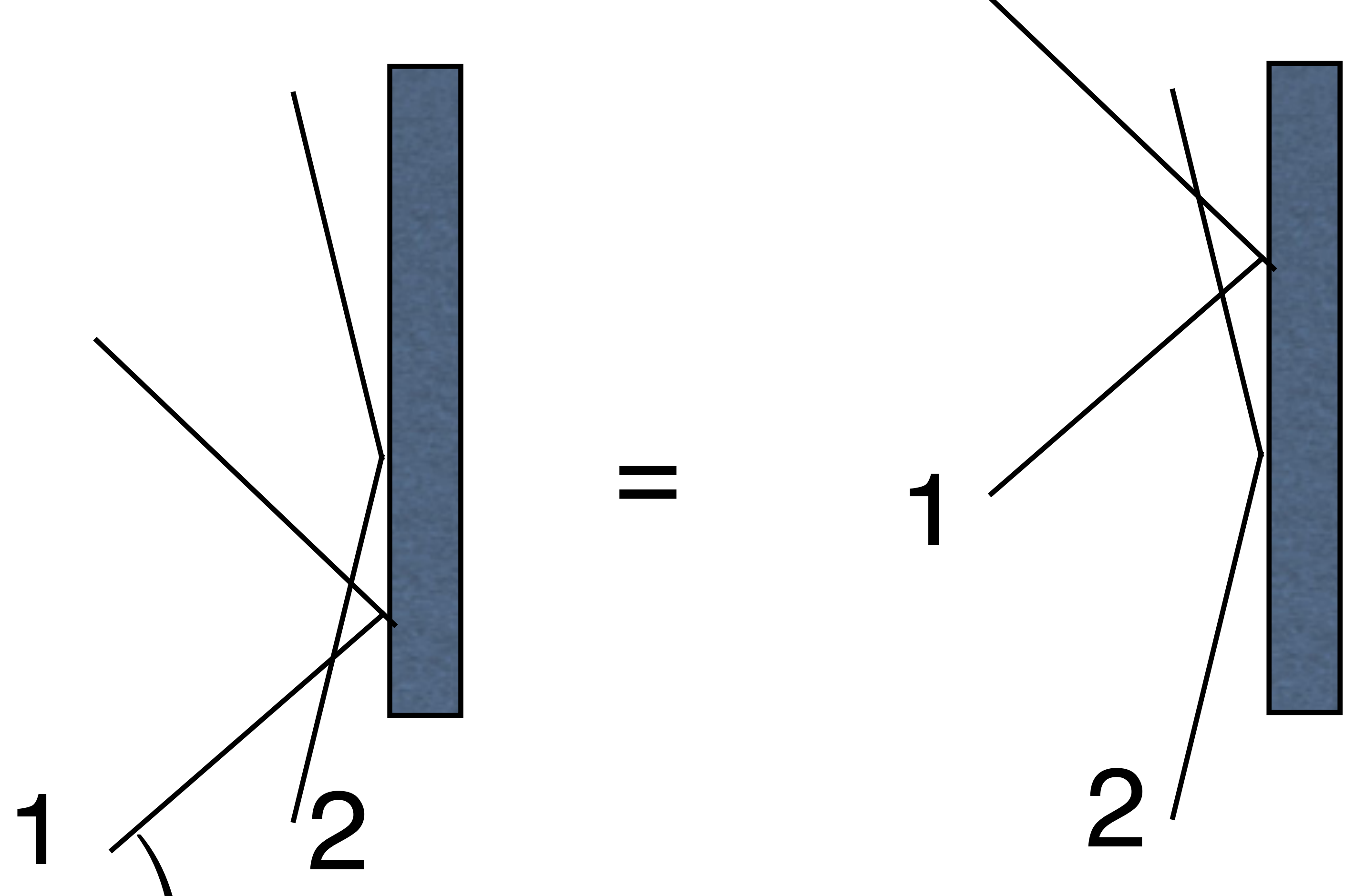
$$p = 0, \ldots, n$$

$$n = 1, 2, \ldots$$

η anisotropy

Boundary Yang-Baxter equation



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[Martins, Guan 2000; RN, Pimenta 2018]

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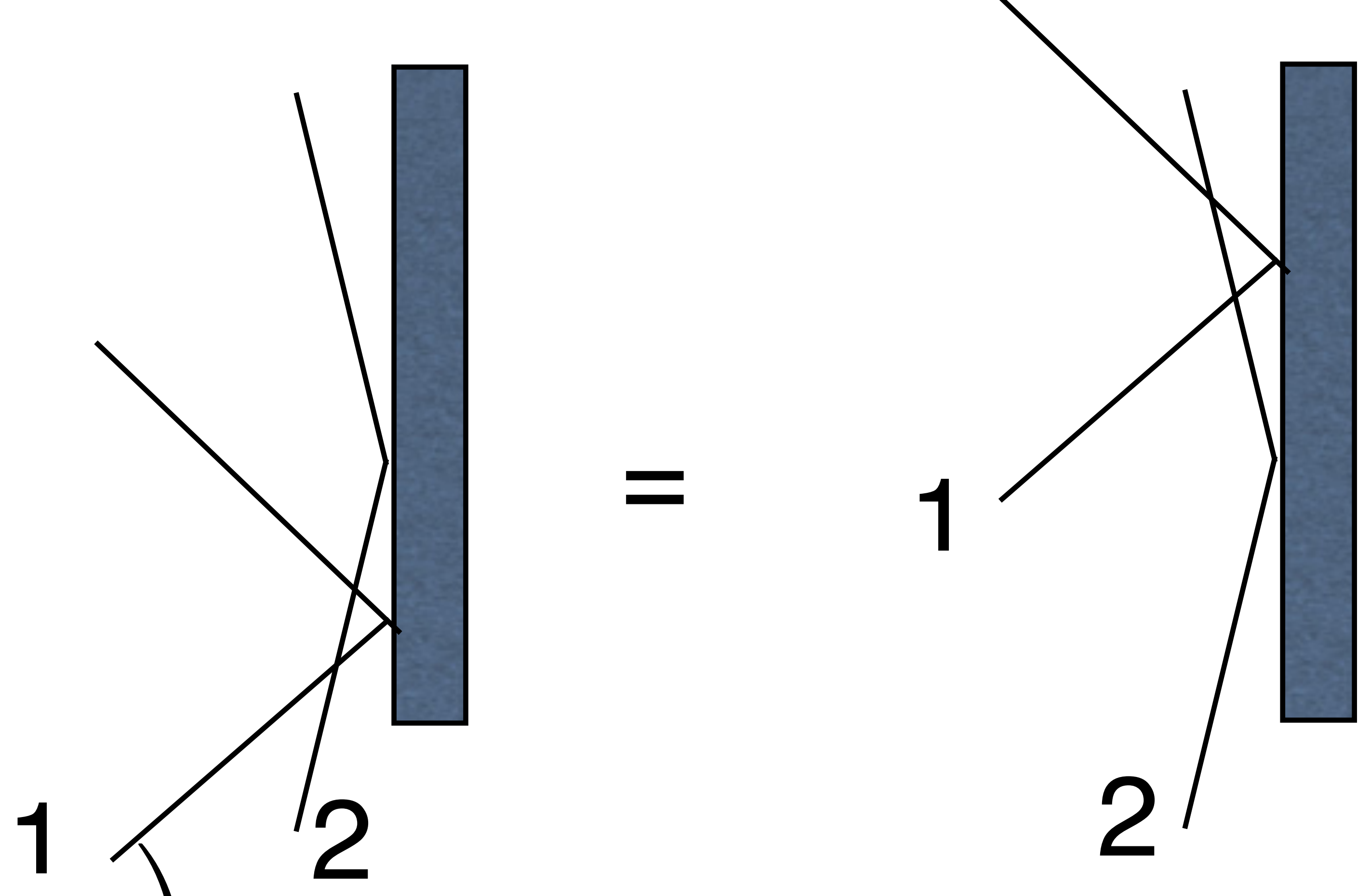
$$p = 0, \ldots, n$$

$$n = 1, 2, \ldots$$

η anisotropy

$$\varepsilon = 0, 1$$

Boundary Yang-Baxter equation



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$$p = 0, \ldots, n$$

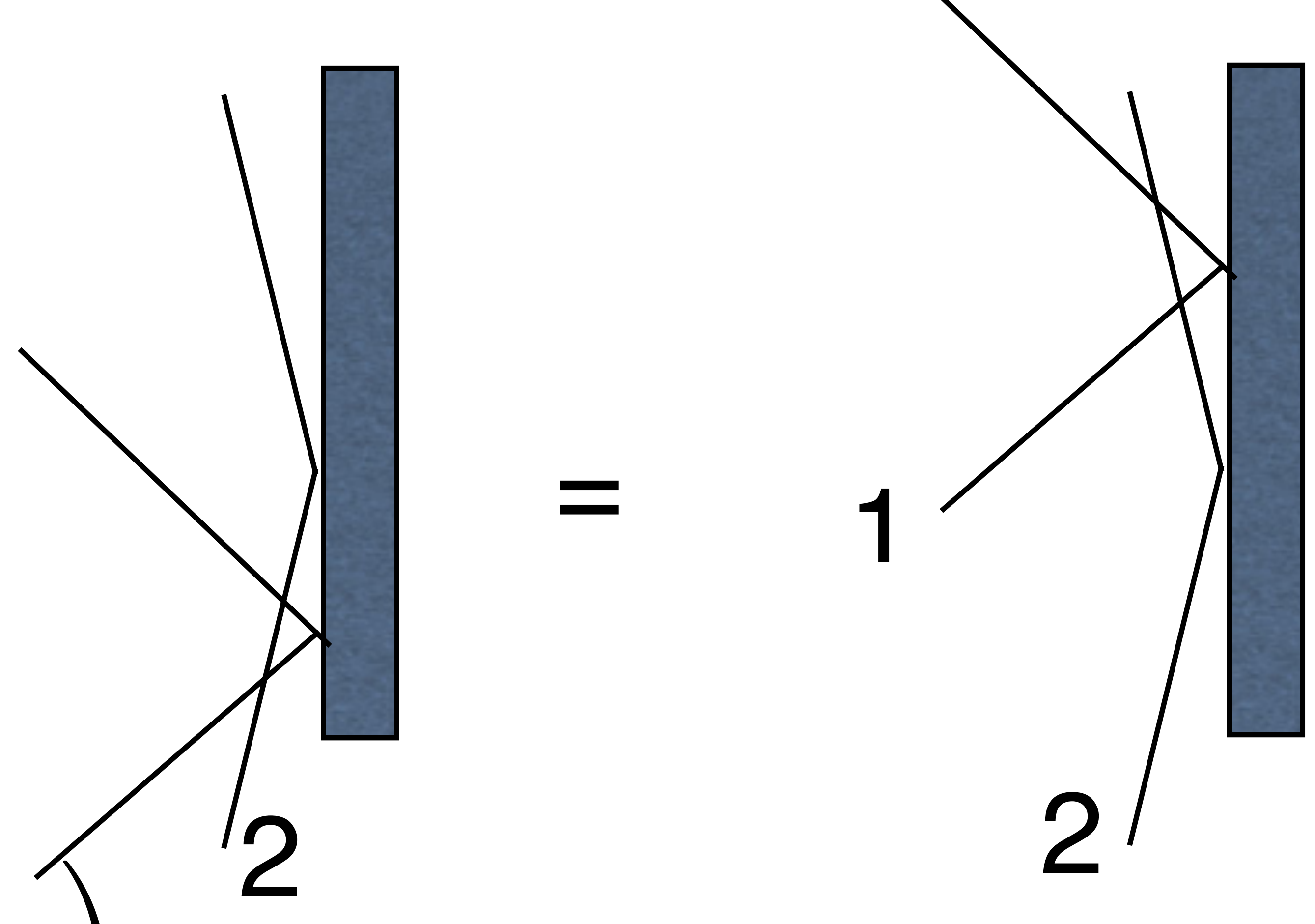
$$n = 1, 2, \ldots$$

η anisotropy

$\varepsilon = \boxed{0}, 1$ [Martins, Guan 2000; RN, Retore 2018]

"conventional" Bethe ansatz

Boundary Yang-Baxter equation



$$\hat{g} = D^{(2)}_{n+1}:$$

$$K^R(u) = \left(\begin{array}{c|c|cc|c|c} k_-(u)\,\mathbb{I}_{p\times p} & & & & & \\ \hline & g(u)\,\mathbb{I}_{(n-p)\times(n-p)} & & & & \\ \hline & & k_1(u) & k_2(u) & & \\ & & k_2(u) & k_1(u) & & \\ \hline & & & & g(u)\,\mathbb{I}_{(n-p)\times(n-p)} & \\ \hline & & & & & k_+(u)\,\mathbb{I}_{p\times p} \end{array}\right)$$

[Martins, Guan 2000; RN, Pimenta 2018]

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$$p = 0, \ldots, n$$

$$n = 1, 2, \ldots$$

η anisotropy

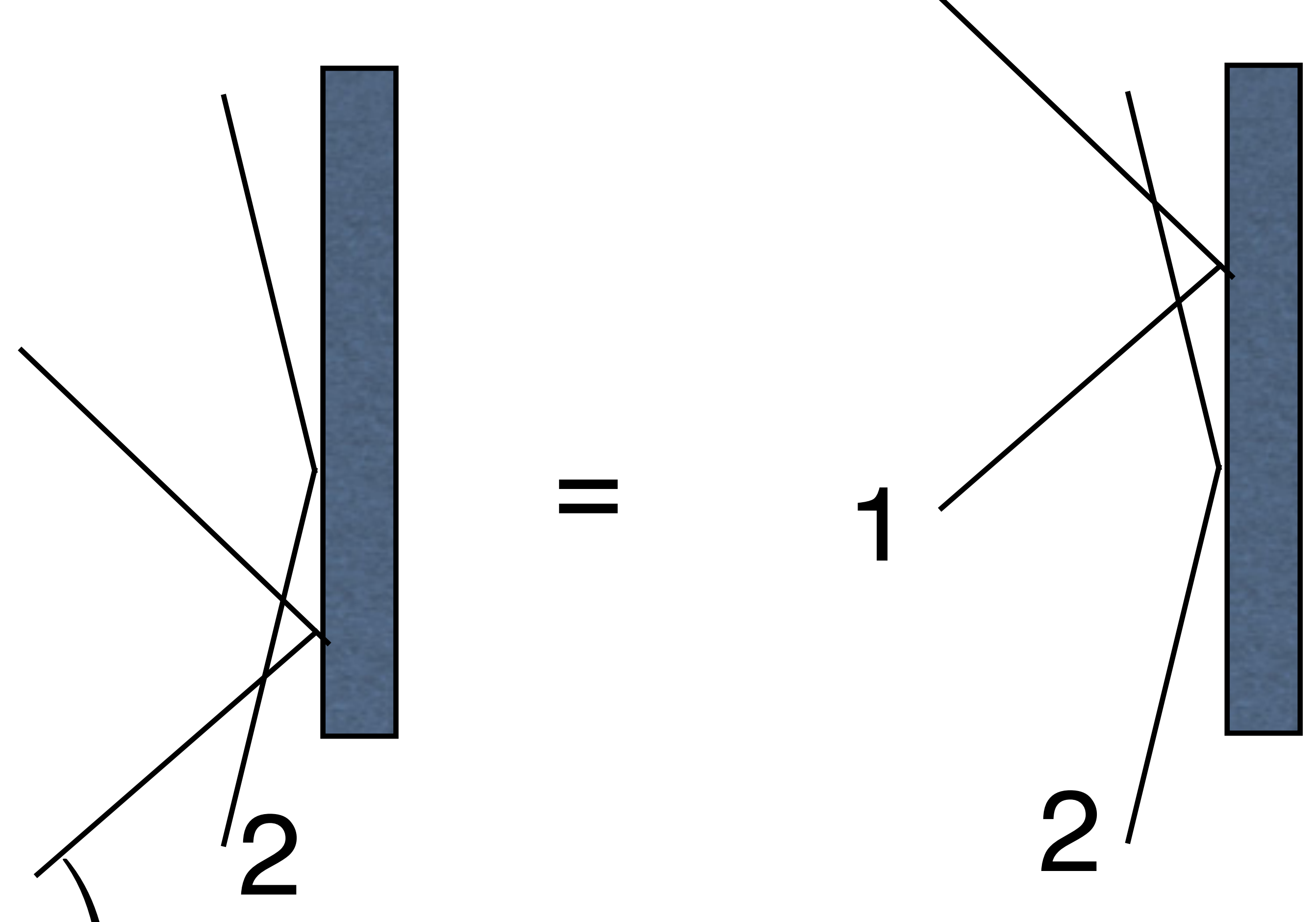
$$\varepsilon = 0, \boxed{1}$$

Bethe ansatz ??

sinh $\longleftrightarrow$ cosh

non-compact CFT!

Boundary Yang-Baxter equation



$\hat{g} = D^{(2)}_{n+1}$:

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 & & k_1(u) & k_2(u) & & \\
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$p = 0, \ldots, n$

$n = \boxed{1}, 2, \ldots$

η anisotropy

$\varepsilon = 0, \boxed{1}$

Bethe ansatz ??

sinh $\longleftrightarrow$ cosh

non-compact CFT!

$\hat{g} = D_2^{(2)} \quad (n=1), \quad \varepsilon = 1 \ :$ QG symmetry $U_q(B_1)$

The problem of finding Bethe ansatz resisted solution for 20 years

- conjectured partial solution [RN, Pimenta, Retore 2019]
- conjectured full solution [Robertson, Jacobsen, Saleur 2020]

cosh instead of sinh!

$$\left(\frac{\sinh(u_j+\eta)}{\sinh(u_j-\eta)}\right)^{2N} = \prod_{k=1;k\neq j}^{m} \frac{\sinh(\frac{1}{2}(u_j-u_k)+\eta)}{\sinh(\frac{1}{2}(u_j-u_k)-\eta)} \boxed{\frac{\cosh(\frac{1}{2}(u_j+u_k)+\eta)}{\cosh(\frac{1}{2}(u_j+u_k)-\eta)}}$$

- proof $so(4) = su(2) \oplus su(2)$ [RN, Retore 2020]

Key idea: $R^{D_2^{(2)}} \sim R^{A_1^{(1)}}\, R^{A_1^{(1)}}\, R^{A_1^{(1)}}\, R^{A_1^{(1)}}$ [Robertson, Pawelkiewicz, Jacobsen, Saleur 2020]

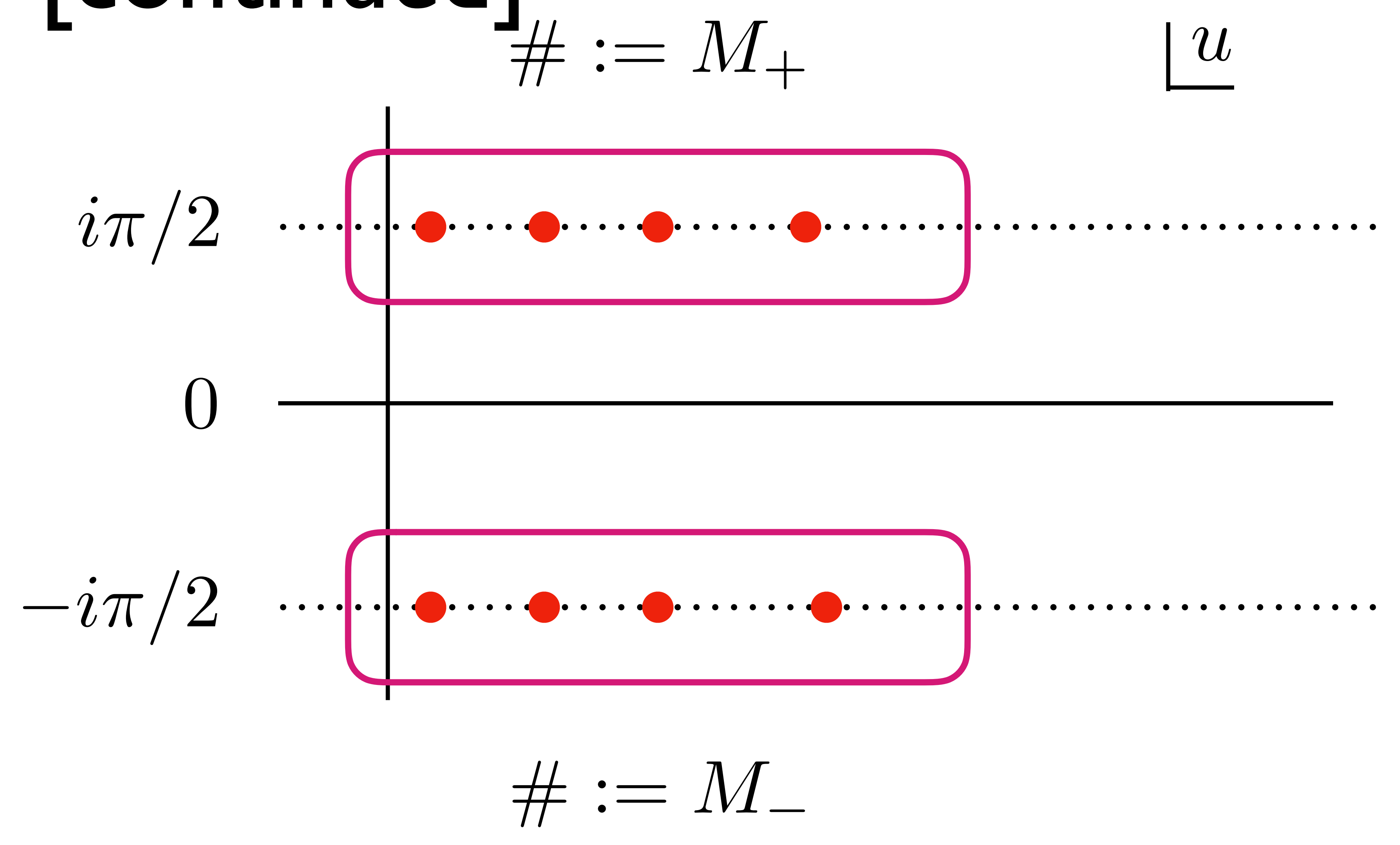
$\Rightarrow \quad t^{D_2^{(2)}} \sim t^{A_1^{(1)}}\, t^{A_1^{(1)}}$ "factorization identity" for transfer matrix

algebraic Bethe ansatz

[Robertson, Jacobsen, Saleur 2020;
Frahm, Gehrmann 2021]

$\hat{g} = D_2^{(2)} \quad (n=1), \quad \varepsilon = 1 \ : \qquad 0 < \eta < \frac{i\pi}{4}$ **critical** **[continued]**

Emergence of a (partially) continuous spectrum:



Ground-state Bethe roots

$$M_+ = M_-$$

There are *excited states* with $\mathtt{n} := M_+ - M_- = 1, 2, \ldots$

$N \to \infty :$

Finite-size scaling: $E(N) \sim N e_\infty + f_\infty + \frac{\pi v_F}{N} \boxed{X_{\text{eff}}}$

effective scaling dimension

$$\boxed{X_{\text{eff}} = [\text{number}] + \mathtt{n}^2 \frac{f(\eta)}{(\log N)^2}}$$

need N ~ 10³ need Bethe ansatz!

Keeping $\mathtt{n}$ fixed as $N \to \infty$ would mean infinite degeneracy in CFT ??

Instead, define limit $\frac{\mathtt{n}}{\log N} \xrightarrow[N\to\infty]{} \boxed{s \in \mathbb{R}_{\geq 0}} \quad \Rightarrow \quad X_{\text{eff}} = [\text{number}] + f(\eta)\, s^2$ continuous

$\hat{g} = D_2^{(2)} \quad (n=1), \quad \varepsilon = 1\ :$ $\qquad 0 < \eta < \frac{i\pi}{4}$ critical [continued]

continuum of conformal dimensions

[Robertson, Jacobsen, Saleur 2020;
Frahm, Gehrmann 2021]

continuum limit is a *non-compact* boundary CFT

does *not* occur for $\varepsilon = 0$

periodic BC:

[Jacobsen, Saleur 2006;
Ikhlef, Jacobsen, Saleur 2008, 2012;
Frahm, Seel 2014]

related to black hole sigma model

[Witten 1991;
Dijkgraaf, Verlinde, Verlinde 1992;
Hanany, Prezas, Troost 2002;
Bazhanov, Kotousov, Koval, Lukyanov 2019, 2020]

$\hat{g} = D_{n+1}^{(2)}, \quad \varepsilon = 1$:

QG symmetry $U_q(B_{n-p}) \otimes U_q(B_p)$ $\quad p = 0, \ldots, n$

analytical Bethe ansatz

[Frahm, Gehrmann, RN, Retore 2509.00610]

$$\left[\frac{\sinh(u_k^{[1]} + \eta)}{\sinh(u_k^{[1]} - \eta)}\right]^{2N} \boxed{\Phi_{1,p,n}(u_k^{[1]})} = \frac{Q_k^{[1]}\left(u_k^{[1]} + 2\eta\right)}{Q_k^{[1]}\left(u_k^{[1]} - 2\eta\right)} \frac{Q_k^{[1]}\left(u_k^{[1]} + 2\eta + i\pi\right)}{Q_k^{[1]}\left(u_k^{[1]} - 2\eta + i\pi\right)} \frac{Q^{[2]}\left(u_k^{[1]} - \eta\right)}{Q^{[2]}\left(u_k^{[1]} + \eta\right)} \frac{Q^{[2]}\left(u_k^{[1]} - \eta + i\pi\right)}{Q^{[2]}\left(u_k^{[1]} + \eta + i\pi\right)}$$

$\neq 1$ only if $l = p$

$$\boxed{\Phi_{l,p,n}(u_k^{[l]})} = \frac{Q^{[l-1]}\left(u_k^{[l]} - \eta\right)}{Q^{[l-1]}\left(u_k^{[l]} + \eta\right)} \frac{Q^{[l-1]}\left(u_k^{[l]} - \eta + i\pi\right)}{Q^{[l-1]}\left(u_k^{[l]} + \eta + i\pi\right)} \frac{Q_k^{[l]}\left(u_k^{[l]} + 2\eta\right)}{Q_k^{[l]}\left(u_k^{[l]} - 2\eta\right)} \frac{Q_k^{[l]}\left(u_k^{[l]} + 2\eta + i\pi\right)}{Q_k^{[l]}\left(u_k^{[l]} - 2\eta + i\pi\right)}$$
$$\times \frac{Q^{[l+1]}\left(u_k^{[l]} - \eta\right)}{Q^{[l+1]}\left(u_k^{[l]} + \eta\right)} \frac{Q^{[l+1]}\left(u_k^{[l]} - \eta + i\pi\right)}{Q^{[l+1]}\left(u_k^{[l]} + \eta + i\pi\right)}, \qquad l = 2, \ldots, n-1$$

$$Q^{[l]}(u) = \prod_{j=1}^{m_l} \sinh\left(\tfrac{1}{2}(u - u_j^{[l]})\right) \sinh\left(\tfrac{1}{2}(u + u_j^{[l]})\right), \qquad l = 1, \ldots, n-1,$$

$$Q^{\boxed{[n]}}(u) = \prod_{j=1}^{m_n} \sinh\left(\tfrac{1}{2}(u - u_j^{[n]})\right) \boxed{\cosh\left(\tfrac{1}{2}(u + u_j^{[n]})\right)}$$

cosh instead of sinh!

continuous spectrum?

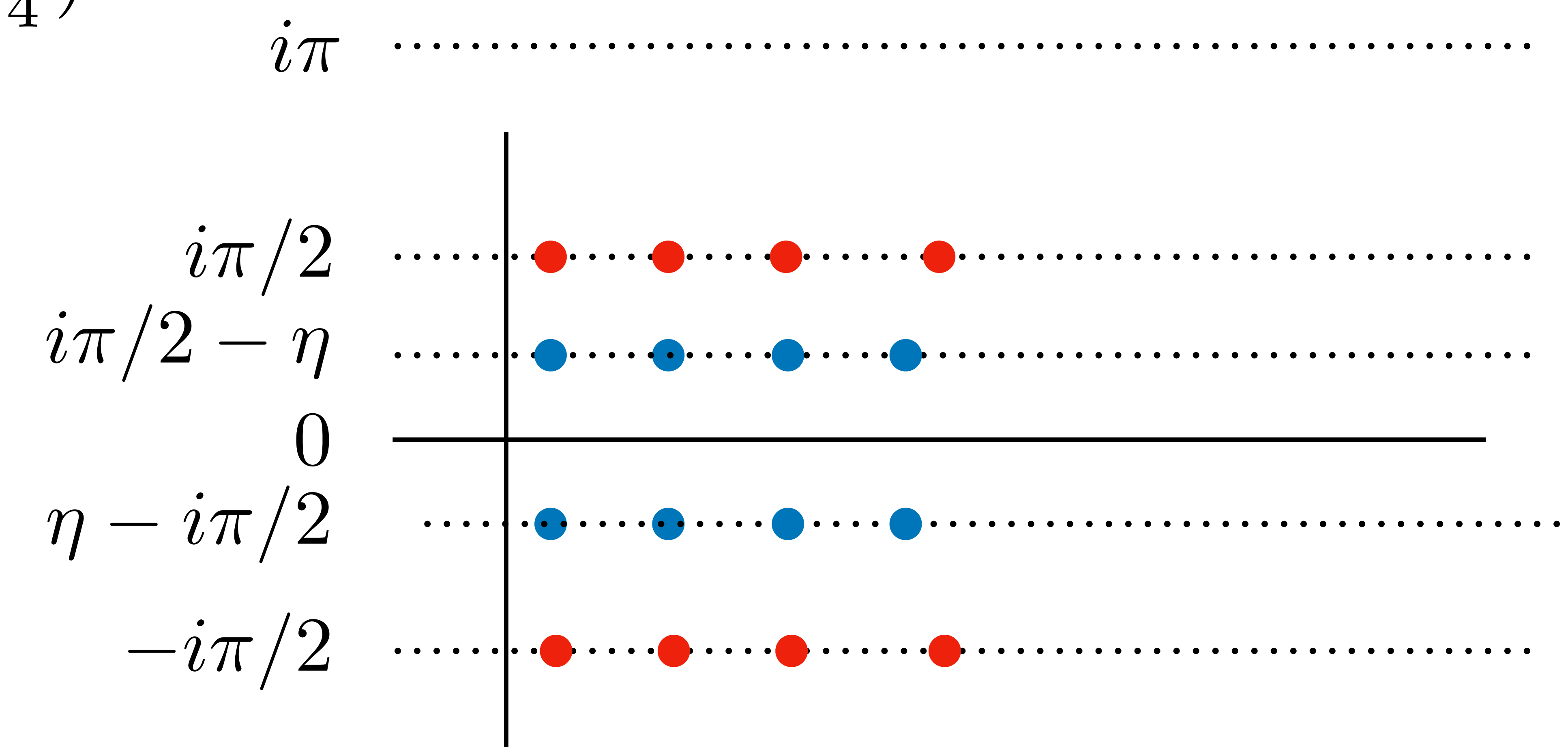
$$\hat{g} = D_3^{(2)} \quad (n = 2), \quad \varepsilon = 1, \quad p = 0, 1 : \quad \eta = i\gamma\,, \quad \gamma \in (0, \tfrac{\pi}{4})$$

Find (partially) continuous spectrum:

level-2

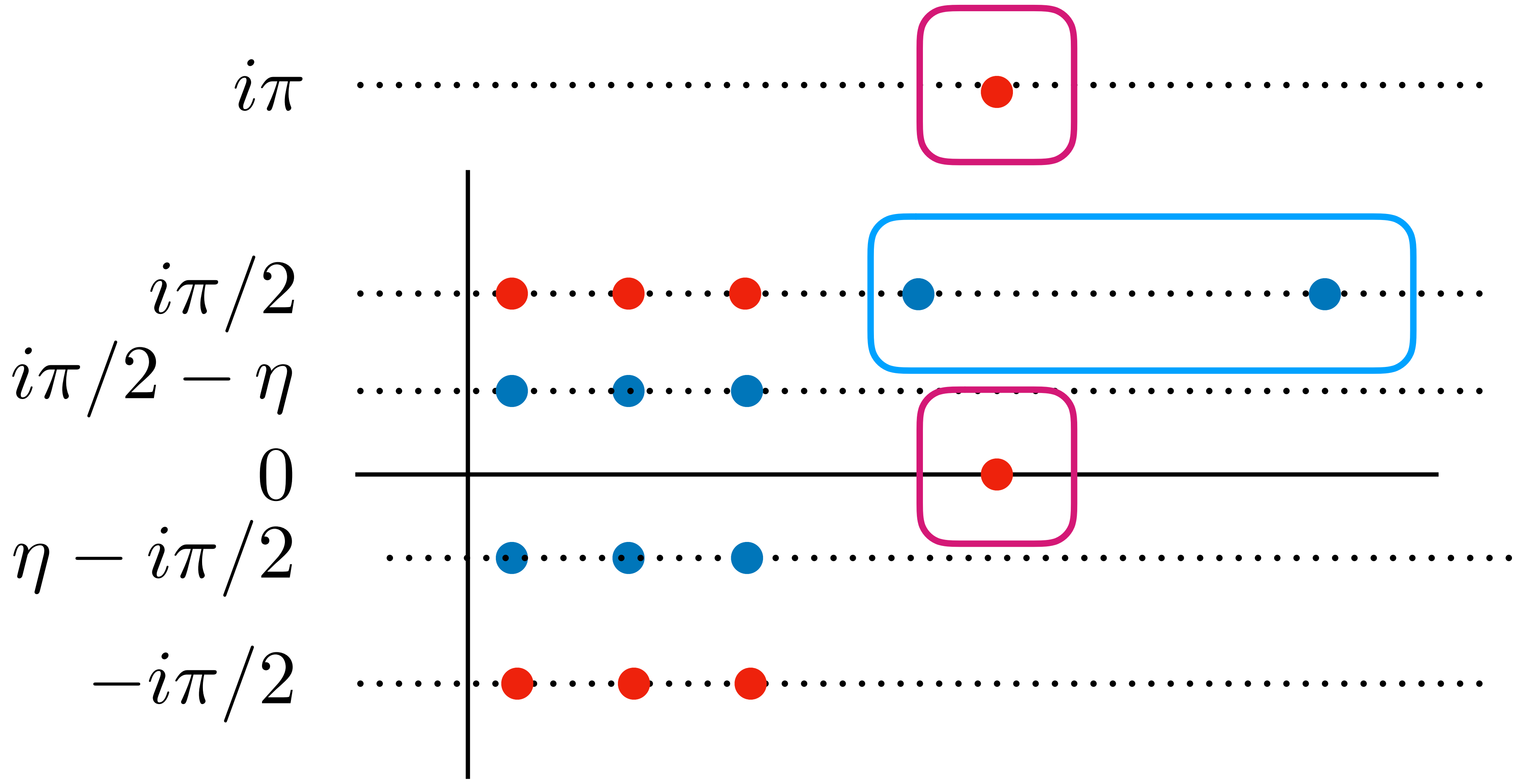
level-1

Ground-state Bethe roots



Excited-state Bethe roots (example)

$\mathtt{n} = 2:$



$$X_{\text{eff}} = [\text{number}] + (2\mathtt{n}+1)^2 \frac{f(\eta)}{(\log N)^2} \qquad \mathtt{n} = 1, 2, 3, \ldots$$

does *not* occur for $\varepsilon = 0$

Outline

I. Review of quantum integrable models with quantum group symmetry

- Quantum integrability (sorry to the experts!)
- Symmetries & degeneracies of integrable spin chains (rank-1 algebra)

  $\longrightarrow$ *quantum group* symmetry

- Generalization to higher-rank algebras
- Bethe ansatz

II. $D^{(2)}$ models

$\boxed{\text{III. Outlook}}$

- analytical Bethe ansatz for $\hat{g} = D^{(2)}_{n+1}, \quad \varepsilon = 1$

- Hamiltonian is not Hermitian

  For $N \to \infty$ all low-energy states have *real* energy E $\therefore$ sensible CFT

- (partially) continuous spectrum for $n = 1, 2$

- conjecture: (partially) continuous spectrum for all $n = 1, 2, \ldots$

- identification of the continuum non-compact CFTs for n>1?

Thank you for your attention!