

The off-diagonal Bethe ansatz method of quantum integrable systems

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Joint Works with J. Cao, K. Shi and Y. Wang

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Challenges in Integrability, Wuhan, [November 15, 2025](#)

- Introduction
- Heisenberg spin chain with the periodic boundary
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- Heisenberg spin chain with general boundary terms
 - T-W relation and root pattern for the ground state.
 - Thermodynamic limit.
- Conclusion and Comments

Quantum integrable systems have many applications in

- String/ Gauge theories: AdS/CFT, Super-symmetric Yang-Mills theories...
- Statistical mechanics: The Ising model, the six-vertex models...
- Condensed Matter Physics: The super-symmetric $t - J$ Model, the Hubbard model...
- Mathematics: Quantum group, Representation theory, Algebraic Topology, ...

I. Introduction

Methods to solve the spectrum

There are many methods to solve quantum integrable systems (The case of $T = 0$):

- The Coordinate Bethe Ansatz method (Bethe 1931)
- The Baxter's $T - Q$ relation method (Baxter 1970s)
- The Quantum Inverse Scattering (or Algebraic Bethe Ansatz) method (Faddeev's School 1979s) and its generalizations (such as the analytic Bethe Ansatz method (Reshetikhin 1983 and Mezincescu & Nepomechie 1992, the separation of variables method (Sklyanin 1985, the Lyon's group 2013), the q-Onsager algebra approach (Baseilhac 2006))
- **The off-diagonal Bethe Ansatz method (Wang's school 2013s)**
- The modified algebraic Bethe Ansatz (Belliard et. al 2013s)

⋮

II. Heisenberg chain with the periodic boundary condition

Eigenvalues relation and their roots

The Hamiltonian of the closed Heisenberg chain is

$$H = \sum_{k=1}^N \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z \right),$$

where

$$\sigma_{N+1}^\alpha = \sigma_1^\alpha, \quad \alpha = x, y, z.$$

The system is **integrable**, i.e., there exist enough conserved charges

$$i\hbar \frac{\partial}{\partial t} h_i = [H, h_i] = 0, \quad i = 1, \dots$$

and

$$[h_i, h_j] = 0.$$

II. Heisenberg chain with the periodic boundary condition

Eigenvalues relation and their roots

It is convenient to introduce a generation function of these charges, the so-called transfer matrix

$$t(u) = \sum_{i=0} h_i u^i = \text{tr} T(u) = A(u) + D(u).$$

Then

$$[t(u), t(v)] = 0, \quad H \propto \frac{\partial}{\partial u} \ln t(u)|_{u=0} + \text{const},$$

or

$$H \propto h_0^{-1} h_1 + \text{const},$$
$$h_0 \sigma_i^\alpha h_0^{-1} = \sigma_{i+1}^\alpha.$$

II. Heisenberg chain with the periodic boundary condition

Eigenvalues relation and their roots

The eigenstates and the corresponding eigenvalues can be obtained by Quantum Inverse Scattering Method (QISM). In the framework of QISM, the monodromy matrix $T(u)$

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

has played a central role. It is built from the six-vertex R-matrix of

$$T_0(u) = R_{0N}(u - \theta_N) \dots R_{01}(u - \theta_1),$$

where the well-known six-vertex R-matrix is given by

$$R(u) = \begin{pmatrix} u + \eta & & & \\ & u & \eta & \\ & \eta & u & \\ & & & u + \eta \end{pmatrix}.$$

The transfer matrix is $t(u) = \text{tr} T(u) = A(u) + D(u)$, where $\eta = \sqrt{-1}$.

II. Heisenberg chain with the periodic boundary condition

Eigenvalues relation and their roots

The R-matrix satisfies the Yang-Baxter equation (YBE)

$$R_{12}(u-v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u-v). \quad (1)$$

The above fundamental relation leads to the following so-called RLL relation between the monodromy matrix

$$R_{00'}(u-v) T_0(u) T_{0'}(v) = T_{0'}(v) T_0(u) R_{00'}(u-v).$$

This leads to

$$[t(u), t(v)] = 0, \quad (2)$$

which ensures the integrability of the Heisenberg chain with periodic boundary condition.

II. Heisenberg chain with the periodic boundary condition

Eigenvalues relation and their roots

Besides the YBE, the R-matrix has the following properties

$$\text{Initial condition : } R_{0,j}(0) = \eta P_{0,j},$$

$$\text{Unitary relation : } R_{0,j}(u) R_{j,0}(-u) = \phi(u) \times \text{id} \otimes \text{id},$$

$$\text{Crossing relation : } R_{0,j}(u) = -\sigma_0^y R_{0,j}^{t_0}(-u - \eta) \sigma_0^y,$$

$$\text{PT-symmetry : } R_{0,j}(u) = R_{j,0}(u) = R_{0,j}^{t_0 t_j}(u),$$

$$\text{Z}_2\text{-symmetry : } \sigma_0^\alpha \sigma_j^\alpha R_{0,j}(u) = R_{0,j}(u) \sigma_0^\alpha \sigma_j^\alpha, \text{ for } \alpha = x, y, z,$$

$$\text{Fusion condition : } R_{0,j}(\pm\eta) = \eta(\pm 1 + P_{0,j}) = \pm 2\eta P_{0,j}^{(\pm)},$$

where $\phi(u) = \eta^2 - u^2$.

By using the fusion technique (Kulish et al 1981, Kirillov et al, 1986), one can derive the relation

$$t(u) t(u - \eta) = a(u) d(u - \eta) \times \text{id} + d(u) \mathbb{W}(u), \quad d(u) = \prod_{j=1}^N (u - \theta_j) = a(u - \eta), \quad (3)$$

where $\mathbb{W}(u)$ is a descendent operator, and $t(u)$ and $\mathbb{W}(u)$ commute with each other,

$$[t(u), t(v)] = [\mathbb{W}(u), \mathbb{W}(v)] = [t(u), \mathbb{W}(v)] = 0. \quad (4)$$

Acting the operators on a common eigenstate $|\Psi\rangle$

$$t(u) |\Psi\rangle = \Lambda(u) |\Psi\rangle, \quad \mathbb{W}(u) |\Psi\rangle = W(u) |\Psi\rangle,$$

II. Heisenberg chain with the periodic boundary condition

Eigenvalues relation and their roots

we have the very relation between $\Lambda(u)$ and $W(u)$, called it as the $t - W$ relation,

$$\Lambda(u) \Lambda(u - \eta) = a(u) d(u - \eta) + d(u) W(u), \quad (5)$$

where the polynomials $\Lambda(u)$ and $W(u)$ with the degree N have decompositions

$$\Lambda(u) = 2 \prod_{j=1}^N (u - z_j + \frac{\eta}{2}), \quad W(u) = 3 \prod_{j=1}^N (u - w_j).$$

Taking $u = \theta_j$ for the $t - W$ relation (5), we have

$$\Lambda(\theta_j) \Lambda(\theta_j - \eta) = a(\theta_j) d(\theta_j - \eta), \quad j = 1, \dots, N. \quad (6)$$

The relations allow us that the eigenvalue $\Lambda(u)$ of the transfer matrix $t(u)$ can be parameterized by some parameters $\{\lambda_1, \dots, \lambda_M | M = 0, \dots, N\}$ as follows (see also the conventional Bethe ansatz methods):

$$\Lambda(u) = a(u) \frac{Q(u - \eta)}{Q(u)} + d(u) \frac{Q(u + \eta)}{Q(u)}, \quad Q(u) = \prod_{j=1}^M (u - \lambda_j),$$

the parameters $\{\lambda_i\}$ should satisfy Bethe ansatz equations,

$$\prod_{k \neq j}^M \frac{\lambda_j - \lambda_k + \eta}{\lambda_j - \lambda_k - \eta} = \prod_{l=1}^N \frac{\lambda_j - \theta_l + \eta}{\lambda_j - \theta_l}, \quad j = 1, \dots, M.$$

II. Heisenberg chain with the periodic boundary condition

Eigenvalues relation and their roots: $T - Q$ relation

$$\text{BAEs} \Rightarrow \Lambda(u) \Rightarrow W(u)$$

The eigenvalue can be given in terms of a homogeneous $T - Q$ relation

$$\Lambda(u) = a(u) \frac{Q(u - \eta)}{Q(u)} + d(u) \frac{Q(u + \eta)}{Q(u)},$$

$$W(u) = a(u) \frac{Q(u - 2\eta)}{Q(u)} + d(u) \frac{Q(u + \eta)Q(u - 2\eta)}{Q(u)Q(u - \eta)} + d(u - \eta) \frac{Q(u + \eta)}{Q(u - \eta)},$$

where the roots of $Q(u)$ satisfy the Bethe ansatz equations (BAEs)

$$\frac{a(\lambda_j)}{d(\lambda_j)} = -\frac{Q(\lambda_j + \eta)}{Q(\lambda_j - \eta)}, \quad j = 1, \dots, M.$$

$$\text{BAEs} \Rightarrow \text{TBA}$$

II. Heisenberg chain with the periodic boundary condition

Eigenvalues relation and their roots

Taking $\{u = z_j - \frac{\eta}{2}\}$, $\{u = w_j\}$ and $\{\theta_j = 0\}$, we have

$$(z_j + \frac{\eta}{2})^N (z_j - \frac{3}{2}\eta)^N = -(z_j - \frac{\eta}{2})^N W(z_j - \frac{\eta}{2}), \quad j = 1, \dots, N, \quad (7)$$

$$\Lambda(w_j) \Lambda(w_j - \eta) = (w_j + \eta)^N (w_j - \eta)^N, \quad j = 1, \dots, N. \quad (8)$$

The above equations allow one to determine the polynomials $\Lambda(u)$ and $W(u)$. Moreover, one can show that

$$\{z_j^*\} = \{z_j\}, \quad \{w_j^*\} = \{w_j\}.$$

II. Heisenberg chain with the periodic boundary condition

Thermodynamic limit

Alternatively, we may consider the root patterns of $\{z_j\}$ and $\{w_j\}$ for some particular states such as the ground state.

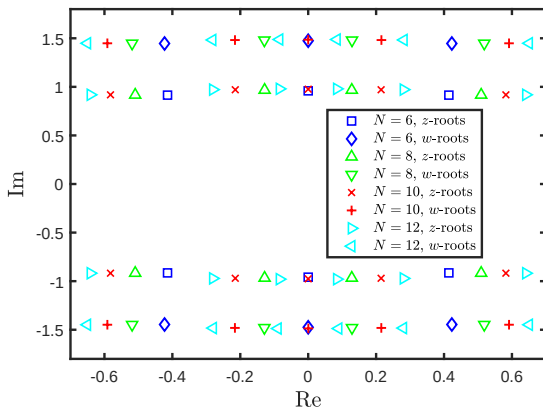


Fig. 1. Patterns of zero roots at the ground state with $N = 6, 8, 10, 12$. The data are obtained by using the exact numerical diagonalization with $\{\theta_j = 0\}$.

II. Heisenberg chain with the periodic boundary condition

Thermodynamic limit

For the ground state, we have

- All the z -roots form conjugate pairs as $\{u_j^{(2)} \pm \eta | j = 1, \dots, N/2\}$ with real $u_j^{(2)}$.
- All the w -roots form conjugate pairs as $\{\bar{u}_j^{(2)} \pm \frac{3\eta}{2} | j = 1, \dots, N/2\}$ with real $\bar{u}_j^{(2)}$.

the corresponding eigenvalues $\Lambda_g(u)$ and $W_g(u)$ can be given as

$$\Lambda_g(u) = 2 \prod_{j=1}^{N/2} (u - u_j^{(2)} - \frac{\eta}{2})(u - u_j^{(2)} + \frac{3\eta}{2}) = e^{N[\lambda_g^{(0)}(u) + \frac{1}{N}\lambda_g^{(1)}(u) + O(\frac{1}{N^2})]},$$

$$W_g(u) = 3 \prod_{j=1}^{N/2} (u - \bar{u}_j^{(2)} - \frac{3}{2}\eta)(u - \bar{u}_j^{(2)} + \frac{3}{2}\eta) = e^{N[w_g^{(0)}(u) + \frac{1}{N}w_g^{(1)}(u) + O(\frac{1}{N^2})]}.$$

In the thermodynamic limit $N \rightarrow \infty$, $u_j^{(2)}$ and $\bar{u}_j^{(2)}$ become dense on the real line and form the densities of $u_j^{(2)}$ and $\bar{u}_j^{(2)}$. The density of the z -roots allow us to rederive

- The ground energy E_g

$$E_g = -2Ni \int_{-\infty}^{\infty} \left(\frac{1}{\lambda + \frac{i}{2}} + \frac{1}{\lambda - \frac{3i}{2}} \right) (\rho_w^{(0)}(\lambda) + \rho_w^{(1)}(\lambda)) d\lambda - N = (1 - 4 \ln 2)N,$$

- The eigenvalues of the transfer matrix for the ground state

$$\Lambda_g(u) = \left(\frac{2\Gamma(1 + \frac{i u}{2})\Gamma(\frac{3}{2} - \frac{i u}{2})}{\Gamma(\frac{1}{2} + \frac{i u}{2})\Gamma(1 - \frac{i u}{2})} \right)^N e^{O(\frac{1}{N})}.$$

II. Heisenberg chain with the periodic boundary condition

Thermodynamic limit

Substituting $z_j = u_j^{(2)} + \frac{\eta}{2}$ and $z_j = u_j^{(2)} - \frac{\eta}{2}$ into (7) respectively, we obtain

$$(u_j^{(2)} + \frac{3\eta}{2})^N (u_j^{(2)} - \frac{\eta}{2})^N = -(u_j^{(2)} + \frac{\eta}{2})^N W_g(u_j^{(2)} + \frac{\eta}{2}), \quad j = 1, \dots, \frac{N}{2},$$

$$(u_j^{(2)} + \frac{\eta}{2})^N (u_j^{(2)} - \frac{3\eta}{2})^N = -(u_j^{(2)} - \frac{\eta}{2})^N W_g(u_j^{(2)} - \frac{\eta}{2}), \quad j = 1, \dots, \frac{N}{2},$$

which implies

$$(u_j^{(2)} + \frac{3\eta}{2})^N (u_j^{(2)} - \frac{3\eta}{2})^N = W_g(u_j^{(2)} + \frac{\eta}{2}) W_g(u_j^{(2)} - \frac{\eta}{2}), \quad j = 1, \dots, \frac{N}{2}.$$

Namely,

$$\frac{1}{N} \frac{\partial}{\partial u} \ln[W_g(u + \frac{\eta}{2}) W_g(u - \frac{\eta}{2})] = \frac{1}{u + \frac{3\eta}{2}} + \frac{1}{u - \frac{3\eta}{2}} + O(\frac{1}{N^2}),$$

$$\frac{1}{N} \ln W_g(0) = \frac{1}{N} \ln 3 + O(\frac{1}{N^2}).$$

II. Heisenberg chain with the periodic boundary condition

Thermodynamic limit

As a consequence, we have

$$\rho_w^{(0)}(\lambda) = \frac{1}{2 \cosh(\pi \lambda)} = \rho_\Lambda^{(0)}(\lambda), \quad \rho_w^{(1)}(\lambda) = 0 = \rho_\Lambda^{(1)}(\lambda),$$

which leads to

$$W_g(u) = 3 \left(\frac{(u + \eta)(u - \eta)}{u} \right)^N \left(\tanh \frac{\pi u}{2} \right)^N e^{O(\frac{1}{N})}. \quad (9)$$

The $t - W$ relation (6) becomes

$$\Lambda_g(u) \Lambda_g(u - \eta) = (u + \eta)^N (u - \eta)^N \left[1 + 3 \left(\tanh \frac{\pi u}{2} \right)^N e^{O(\frac{1}{N})} \right]. \quad (10)$$

This gives rise to the inverse relation

$$\Lambda_g(u) \Lambda_g(u - \eta) = (u + \eta)^N (u - \eta)^N \left[1 + e^{-\delta N} \right], \quad \text{for a positive } \delta.$$

III. Heisenberg spin chain with general boundary terms

T-W relation and root pattern for the ground state

The Hamiltonian of the Heisenberg chain with unparallel boundary fields is

$$H = \sum_{k=1}^{N-1} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z \right) + \frac{\eta}{p} \sigma_1^z + \frac{\eta}{q} (\sigma_N^z + \xi \sigma_N^x). \quad (11)$$

The system is **integrable**, i.e., the corresponding transfer matrix $t(u)$ can be constructed by the R-matrix and the associated K-matrices

$$t(u) = \text{tr}(K^+(u) \mathcal{T}(u)) = \text{tr}(K^+(u) T(u) K^-(u) T^{-1}(-u)),$$

where the K-matrices $K^\pm(u)$ are the diagonal K-matrices

$$K^-(u) = \begin{pmatrix} p+u & \\ & p-u \end{pmatrix}, \quad K^+(u) = \begin{pmatrix} q+u+\eta & \xi(u+\eta) \\ \xi(u+\eta) & q-u-\eta \end{pmatrix},$$

with the boundary parameters

$$p^* = -p, \quad q^* = -q, \quad \xi^* = \xi.$$

The Hamiltonian can be given in terms of the transfer matrix

$$H = \eta \frac{\partial}{\partial u} \ln t(u)|_{u=0, \{\theta_j\}=0} - N.$$

III. Heisenberg spin chain with general boundary terms

T-W relation and root pattern for the ground state

Following the similar fusion technique, we can derive the relation of the transfer matrices

$$t(u) t(u - \eta) = \frac{\Delta(u) \times \text{id}}{(u + \frac{\eta}{2})(u - \frac{\eta}{2})} + \frac{u^2 \bar{d}(u)}{(u + \frac{\eta}{2})(u - \frac{\eta}{2})} \mathcal{W}(u), \quad \Delta(u) = a(u)d(u - \eta).$$

where

$$a(u) = (u + \eta)(u + p)(\sqrt{1 + \xi^2} u + q) \prod_{j=1}^N (u - \theta_j + \eta)(u + \theta_j + \eta),$$

$$d(u) = u(u - p + \eta)(\sqrt{1 + \xi^2}(u + \eta) - q) \prod_{j=1}^N (u - \theta_j)(u + \theta_j),$$

$$\bar{d}(u) = \prod_{j=1}^N (u - \theta_j)(u + \theta_j).$$

The associated transfer matrices $t(u)$ and $\mathcal{W}(u)$ commute with each other,

$$[t(u), t(v)] = [\mathcal{W}(u), \mathcal{W}(v)] = [t(u), \mathcal{W}(v)] = 0.$$

III. Heisenberg spin chain with general boundary terms

T-W relation and root pattern for the ground state

Denote the corresponding eigenvalues of the transfer matrices by $\bar{\Lambda}(u)$ and $\bar{W}(u)$, we have

$$\Delta(u) - \left(u + \frac{\eta}{2}\right) \left(u - \frac{\eta}{2}\right) \bar{\Lambda}(u) \bar{\Lambda}(u - \eta) = u^2 \prod_{j=1}^N (u - \theta_j)(u + \theta_j) \bar{W}(u), \quad (12)$$

where $\bar{\Lambda}(u)$ (or $\bar{W}(u)$) is a polynomial of u with degree $2N + 2$ (or $2N + 4$). It implies that

$$\bar{\Lambda}(\pm\theta_j) \bar{\Lambda}(\pm\theta_j - \eta) = \frac{4\Delta(\pm\theta_j)}{(2\theta_j - \eta)(2\theta_j + \eta)}, \quad j = 1, \dots, N.$$

Combining some properties of eigenvalue (J. Cao et al, Nucl. Phys. B 875 (2013), 152-165):

$$\begin{aligned} \Lambda(-u - \eta) &= \Lambda(u), \\ \Lambda(0) &= 2pq \prod_{l=1}^N (1 - \theta_l)(1 + \theta_l), \\ \lim_{u \rightarrow \infty} \Lambda(u) &= 2u^{2N+2} + \dots, \end{aligned}$$

III. Heisenberg spin chain with general boundary terms

T-W relation and root pattern for the ground state

the eigenvalue $\Lambda(u)$ of the corresponding transfer matrix can be parameterized in terms of an inhomogeneous $T - Q$ relation

$$\begin{aligned}\Lambda(u) = & \bar{a}(u) \frac{Q(u-\eta)}{Q(u)} + \bar{d}(u) \frac{Q(u+\eta)}{Q(u)} \\ & + 2 \left[1 - (1 + \xi^2)^{\frac{1}{2}} \right] u(u+\eta) \frac{\bar{a}(u)\bar{d}(u)}{Q(u)},\end{aligned}\quad (13)$$

where

$$\begin{aligned}\bar{a}(u) &= \frac{2(u+\eta)}{2u+\eta} (u+p) [(1+\xi^2)^{\frac{1}{2}}u+q] \prod_{l=1}^N (u-\theta_l+\eta)(u+\theta_l+\eta), \\ \bar{d}(u) &= \bar{a}(-u-\eta), \quad Q(u) = \prod_{j=1}^N (u-\lambda_j)(u+\lambda_j+\eta).\end{aligned}$$

The roots of $Q(u)$ satisfy the BAEs

$$\frac{\bar{a}(\lambda_j)}{\bar{d}(\lambda_j)} + \frac{Q(\lambda_j+\eta)}{Q(\lambda_j-\eta)} = -2[1 - (1 + \xi^2)^{\frac{1}{2}}] \lambda_j(\lambda_j+\eta) \frac{\bar{a}(\lambda_j)}{Q(\lambda_j-\eta)}, \quad j = 1, \dots, N. \quad (14)$$

III. Heisenberg spin chain with general boundary terms

T-W relation and root pattern for the ground state

In the other way, we can parameterize $\bar{\Lambda}(u)$ (or $\bar{W}(u)$) in terms of its roots :

$$\bar{\Lambda}(u) = 2 \prod_{j=1}^{N+1} (u - z_j + \frac{\eta}{2})(u + z_j + \frac{\eta}{2}), \quad \bar{\Lambda}(-u - \eta) = \bar{\Lambda}(u),$$

$$\bar{W}(u) = (\xi^2 - 3) \prod_{k=1}^{N+2} (u - w_k)(u + w_k), \quad \bar{W}(-u) = \bar{W}(u).$$

The relation (12) implies that the above roots satisfy the equations:

$$\Delta(z_j - \frac{\eta}{2}) = (z_j - \frac{\eta}{2})^{2N+2} \bar{W}(z_j - \frac{\eta}{2}), \quad j = 1, \dots, N+1, \quad (15)$$

$$\Delta(w_k) = (w_k + \frac{\eta}{2})(w_k - \frac{\eta}{2}) \bar{\Lambda}(w_k) \bar{\Lambda}(w_k - \eta), \quad k = 1, \dots, N+2. \quad (16)$$

III. Heisenberg spin chain with general boundary terms

T-W relation and root pattern for the ground state

Let us consider the root patterns of $\{z_j\}$ and $\{w_j\}$ for the ground state.

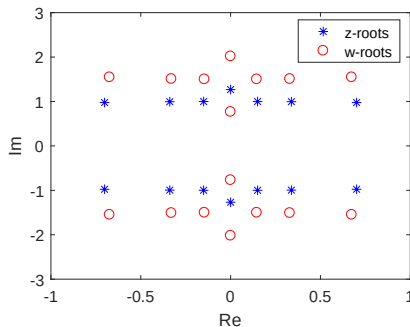


Fig. 2. The patterns of z-roots and w-roots in complex plane at the ground state with $N = 6$, $\eta = i$, $p = -1.2i$, $\bar{q} = 0.8i$, $\xi = 1$. The data are obtained by the exact numerical diagonalization. The blue asterisks indicate z-roots $\{z_j + \frac{\eta}{2}\}$ and red circles represent the w-roots $\{w_j\}$ with the inhomogeneous parameters $\{\theta_j = 0\}$.

III. Heisenberg spin chain with general boundary terms

Thermodynamic limit

- The z -roots form conjugate pairs as $\{\pm z_1\eta, u_j^{(2)} \pm \eta | j = 1, \dots, N\}$.
- The w -roots form conjugate pairs as $\{\pm \chi_1\eta, \pm \chi_2\eta, w_j^{(2)} \pm \frac{3\eta}{2} | j = 1, \dots, N\}$.

the corresponding eigenvalues $\bar{\Lambda}_g(u)$ and $\bar{W}_g(u)$ can be given as

$$\begin{aligned}\bar{\Lambda}_g(u) &= 2(u - (z_1 - \frac{1}{2})\eta)(u + (z_1 + \frac{1}{2})\eta) \prod_{j=1}^{N/2} (u - u_j^{(2)} - \frac{\eta}{2})(u + u_j^{(2)} - \frac{\eta}{2}) \\ &\quad \times (u - u_j^{(2)} + \frac{3\eta}{2})(u + u_j^{(2)} + \frac{3\eta}{2}) \\ &\approx 2(u - (z_1 - \frac{1}{2})\eta)(u + (z_1 + \frac{1}{2})\eta) e^{2N(\bar{\lambda}_g^{(0)}(u) + \frac{1}{2N}\bar{\lambda}_g^{(1)}(u) + O(\frac{1}{N^2}))},\end{aligned}$$

$$\begin{aligned}\bar{W}_g(u) &= (\xi^2 - 3)(u - \chi_1\eta)(u + \chi_1\eta)(u - \chi_2\eta)(u + \chi_2\eta) \prod_{j=1}^{N/2} (u - w_j^{(2)} - \frac{3}{2}\eta)(u + w_j^{(2)} - \frac{3}{2}\eta) \\ &\quad \times (u - w_j^{(2)} + \frac{3}{2}\eta)(u + w_j^{(2)} + \frac{3}{2}\eta) \\ &\approx (\xi^2 - 3)(u - \chi_1\eta)(u + \chi_1\eta)(u - \chi_2\eta)(u + \chi_2\eta) e^{2N(\bar{\omega}_g^{(0)}(u) + \frac{1}{2N}\bar{\omega}_g^{(1)}(u) + O(\frac{1}{N^2}))},\end{aligned}$$

$$\bar{\Lambda}_g(0) \approx 2(z_1 - \frac{1}{2})(z_1 + \frac{1}{2}) e^{2N(\bar{\lambda}_g^{(0)}(u) + \frac{1}{2N}\bar{\lambda}_g^{(1)}(u) + O(\frac{1}{N^2}))} = 2pq \equiv 2p\bar{q}\sqrt{1 + \xi^2},$$

III. Heisenberg spin chain with general boundary terms

Thermodynamic limit

$$\begin{aligned}\bar{\Lambda}_g(u) &= \frac{8\sqrt{1+\xi^2}}{u + \frac{\eta}{2}} \frac{\cosh(\frac{\pi u}{2} - \frac{i\pi}{4})}{\sinh(\frac{\pi u}{2} - \frac{i\pi}{4})} \frac{\Gamma(1 + \frac{i u}{2})\Gamma(\frac{3}{2} - \frac{i u}{2})}{\Gamma(\frac{1}{2} + \frac{i u}{2})\Gamma(1 - \frac{i u}{2})} \frac{\Gamma(\frac{p+1}{2} + \frac{i u}{2})\Gamma(\frac{p+2}{2} - \frac{i u}{2})}{\Gamma(\frac{p}{2} + \frac{i u}{2})\Gamma(\frac{p+1}{2} - \frac{i u}{2})} \\ &\times \frac{\Gamma(\frac{\bar{q}+1}{2} + \frac{i u}{2})\Gamma(\frac{\bar{q}+2}{2} - \frac{i u}{2})}{\Gamma(\frac{\bar{q}}{2} + \frac{i u}{2})\Gamma(\frac{\bar{q}+1}{2} - \frac{i u}{2})} \left(\frac{2\Gamma(1 + \frac{i u}{2})\Gamma(\frac{3}{2} - \frac{i u}{2})}{\Gamma(\frac{1}{2} + \frac{i u}{2})\Gamma(1 - \frac{i u}{2})} \right)^{2N} e^{O(\frac{1}{N})},\end{aligned}\quad (17)$$

$$\begin{aligned}\bar{W}_g(u) &= (\xi^2 - 3)(u - p\eta)(u + p\eta)(u - \bar{q}\eta)(u + \bar{q}\eta) \tanh^2 \frac{\pi u}{2} \\ &\times \frac{(u + \eta)^{2N+1}(u - \eta)^{2N+1}}{u^{2N+2}} \left(\tanh \frac{\pi u}{2} \right)^{2N} e^{O(\frac{1}{N})},\end{aligned}$$

which leads to the relation

$$\begin{aligned}(u + \frac{\eta}{2})(u - \frac{\eta}{2})\bar{\Lambda}_g(u)\bar{\Lambda}_g(u - \eta) &= (1 + \xi^2)(u - p\eta)(u + p\eta)(u - \bar{q}\eta)(u + \bar{q}\eta) \\ &\times (u - \eta)^{2N+1}(u + \eta)^{2N+1} \left\{ 1 - \frac{(\xi^2 - 3)}{1 + \xi^2} \left(\tanh \frac{\pi u}{2} \right)^{2N+2} e^{O(\frac{1}{N})} \right\}.\end{aligned}\quad (18)$$

So far, we have used an unified method to solve the eigenvalue of the ground state for quantum integrable spin chain with/without $U(1)$ -symmetry:

- The spin- $\frac{1}{2}$ Heisenberg chain with the periodic boundary condition.
- The spin- $\frac{1}{2}$ Heisenberg chain with arbitrary boundary fields.
- The open spin chains with general boundary condition associated with the other algebras.
- The super-symmetric t-J model with unparallel boundary fields.
- The Hubbard model with unparallel boundary fields.

Thanks for your attentions