Challenges in Integrability @ Wuhan, China
2025/11/20

Integrable SYK-like models

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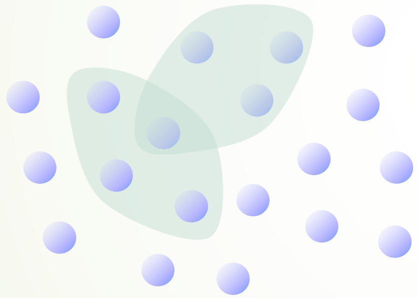


With Kohei Fukai (UTokyo),
Soshun Ozaki (UTokyo→MPI-FKF, Stuttgart)

- S. Ozaki and H. Katsura, Phys. Rev. Research **7**, 013092 (2025)
- K. Fukai and H. Katsura, arXiv:2511.03460
- E. Iyoda, H. Katsura, and T. Sagawa, Phys. Rev. D **98**, 086020 (2018)

Outline

1. Majorana fermion models
   - Introduction & Motivation
   - Clean Majorana SYK
   - Clean supersymmetric Majorana SYK

$$H_4 = -\sum_{i<j<k<l} \gamma_i\gamma_j\gamma_k\gamma_l$$

$$Q_3 = \mathrm{i}\sum_{i<j<k} \gamma_i\gamma_j\gamma_k$$

2. Static and dynamical properties of H_4
   - Finite-T entropy & spectral form factor
   - Out-of-time order correlator (OTOC)
3. Complex fermion models
   - Clean complex SYK
   - Clean supersymmetric complex SYK

$$H_{c4} = \sum_{j<i}\sum_{k<l} c_i^\dagger c_j^\dagger c_k c_l$$

$$Q_{c3} = \sum_{i<j<k} c_i c_j c_k$$

4. Summary

Fermion operators

- **Majorana fermions** $\gamma_i \quad (i = 1, 2, \ldots, N)$ Assume even *N*
  - Defining relations
    - Their own Hermitian conjugate
    - Mutually anti-commuting

$$\gamma_i^\dagger = \gamma_i$$
$$\{\gamma_i, \gamma_j\} = 2\delta_{i,j}$$

  - Explicit representation

$$\gamma_{2j-1} = \left(\prod_{\ell<j} \sigma_\ell^z\right) \sigma_j^x, \qquad \gamma_{2j} = \left(\prod_{\ell<j} \sigma_\ell^z\right) \sigma_j^y$$

Ex.)

$$\gamma_1 = \sigma_1^x, \qquad \gamma_2 = \sigma_1^y$$
$$\gamma_3 = \sigma_1^z \sigma_2^x, \quad \gamma_4 = \sigma_1^z \sigma_2^y$$

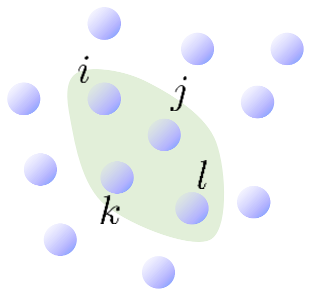
    - Act on a spin chain of length *L*=*N*/2

- **Complex (spinless) fermions**
  - Creation and annihilation operators $c_i^\dagger, c_j \quad (i, j = 1, 2, \ldots, L)$
  - Defining relations $\{c_i, c_j^\dagger\} = \delta_{i,j}, \quad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$
  - Majorana rep. $c_j^\dagger = \frac{1}{2}(\gamma_{2j-1} - \mathrm{i}\gamma_{2j}), \quad c_j = \frac{1}{2}(\gamma_{2j-1} + \mathrm{i}\gamma_{2j})$

Majorana SYK model

- Sachdev-Ye-Kitaev model
  - Hamiltonian

$$H_{\mathrm{SYK}} = \sum_{1\le i<j<k<l\le N} J_{ijkl}\,\gamma_i\gamma_j\gamma_k\gamma_l$$



  - $J_{ijkl} \in \mathbb{R}$: Gaussian with zero mean and variance $J/N^{3/2}$
  - Consists of all-to-all coupling terms
  - Tractable in the large-*N* limit
  - Toy model for holographic duality
  - Maximally chaotic (OTOC saturates the chaos bound)

Sachdev & Ye, PRL **70** (1993), arXiv:cond-mat/**92**12030; Kitaev, Talks at KITP (2015); Maldacena, Stanford, PRD **94** (2016)

- What if all the couplings are equal?
  - Turns out to be *integrable!*
  - Can we still find a signature/precursor of chaos?
  - YES! OTOC shows exponential growth at early times.

Clean Majorana SYK model

Lau, Ma, Murugan & Tezuka, JPA **54**, 095401 (2021)

■ Hamiltonian

$$H_4 = -\sum_{1\le i<j<k<l\le N} \gamma_i\gamma_j\gamma_k\gamma_l$$

- N=4 $\quad H_4 = -\gamma_1\gamma_2\gamma_3\gamma_4 \;= \sigma_1^z\sigma_2^z$
- 2-site Ising model! Trivially integrable.
- H_4 with N > 4 does not seem integrable…

Majorana → spins

$\gamma_1 = \sigma_1^x, \qquad \gamma_2 = \sigma_1^y$

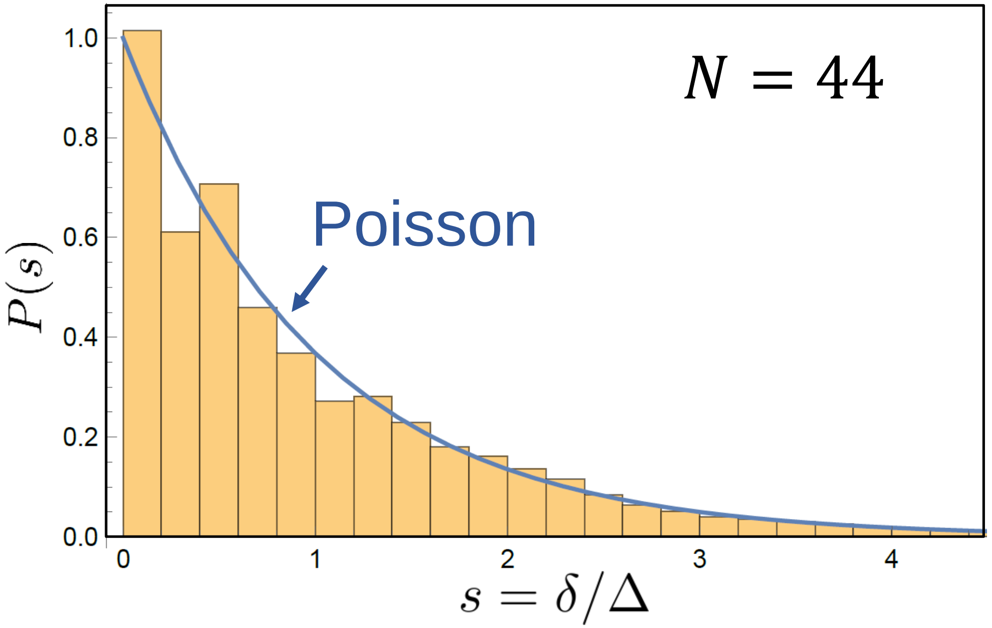
$\gamma_3 = \sigma_1^z\sigma_2^x, \quad \gamma_4 = \sigma_1^z\sigma_2^y$

■ Level-spacing distribution

Casati *et al*, PRL **54** (1985), Pal & Huse, PRB **82** (2010)

- Collect the energy levels in the middle of the spectrum $\quad E_j - E_{\rm GS} \in [0.4N^2, 0.5N^2]$
- Adjacent gaps $\quad \delta_j = E_j - E_{j-1}$
- Poisson distribution!

Is H_4 integrable?? YES!

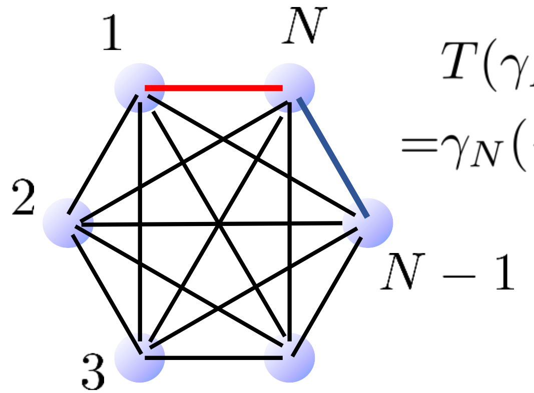


Integrability of quadratic case: warm-up

■Quadratic all-to-all Hamiltonian (N: even)

$$H_2 = \mathrm{i} \sum_{1 \le i < j \le N} \gamma_i \gamma_j$$

Lau *et al*, JPA **54** (2021)



$$T(\gamma_{N-1}\gamma_N)T^{-1} = \gamma_N(-\gamma_1) = \gamma_1\gamma_N$$

- Subject to twisted boundary conditions:

$$T\gamma_j T^{-1} = \gamma_{j+1} \text{ if } 1 \le j < N; \quad T\gamma_N T^{-1} = -\gamma_1$$

- Fourier transform (k = 1, 2, …, N/2)

$$f_k = \frac{1}{\sqrt{2N}} \sum_{j=1}^{N} e^{\mathrm{i}(j-1)\theta_k} \gamma_j, \quad f_k^\dagger = \frac{1}{\sqrt{2N}} \sum_{j=1}^{N} e^{-\mathrm{i}(j-1)\theta_k} \gamma_j$$

θ_k's are determined so that

$$T f_k^{(\dagger)} T^{-1} = e^{\pm \mathrm{i}\theta_k} f_k^{(\dagger)} \Rightarrow \theta_k = \frac{(2k-1)\pi}{N}$$

- They are complex fermions obeying

$$\{f_k, f_\ell^\dagger\} = \delta_{k,\ell}, \quad \{f_k, f_\ell\} = \{f_k^\dagger, f_\ell^\dagger\} = 0$$

- Diagonal form of H_2

$$H_2 = \sum_{k=1}^{N/2} \epsilon_k \left(f_k^\dagger f_k - \frac{1}{2} \right)$$

$$\epsilon_k = 2 \cot \frac{\theta_k}{2}$$

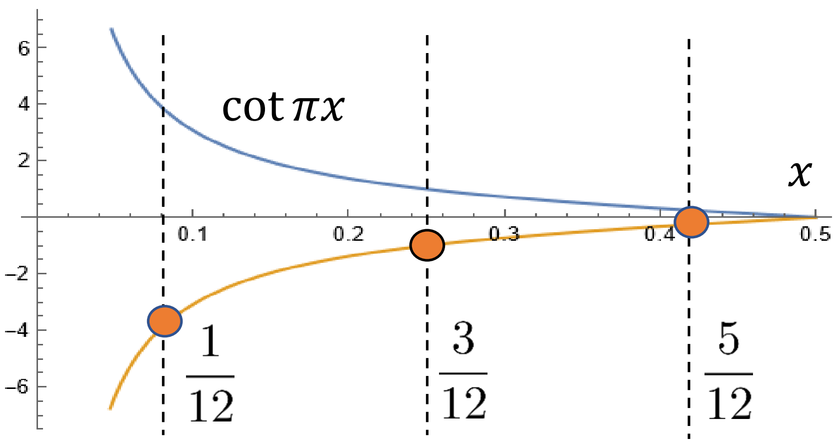
Eigenstates of H_2

■ Dispersion relation

$$H_2 = \sum_{k=1}^{N/2} \epsilon_k \left(f_k^\dagger f_k - \frac{1}{2} \right), \quad \epsilon_k = 2\cot\frac{(2k-1)\pi}{2N}$$

- Eigen-operator: $[H_2, f_k^\dagger] = \epsilon_k f_k^\dagger$

$N = 6$ case $|0,0,0\rangle$



■ Many-body eigenstates

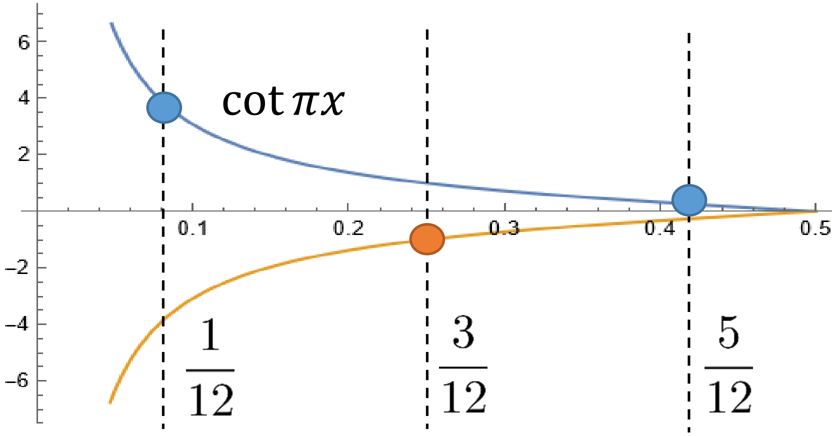
- Vacuum state: $|\mathrm{vac}\rangle$ s.t. $f_k|\mathrm{vac}\rangle = 0\ \forall k$ and $\langle \mathrm{vac}|\mathrm{vac}\rangle = 1$
- Fock states

$$|n_1, n_2, \ldots, n_{N/2}\rangle = (f_1^\dagger)^{n_1}(f_2^\dagger)^{n_2}\cdots(f_{N/2}^\dagger)^{n_{N/2}}|\mathrm{vac}\rangle$$

- Occupation numbers: $n_k = 0$ or 1
- Eigen-energies of H_2

$$E_2(n_1, n_2, \ldots, n_{N/2}) = -\sum_{k=1}^{N/2}(-1)^{n_k}\frac{\epsilon_k}{2}$$

$N = 6$ case $|1,0,1\rangle$



Integrability of $H_4 = -\sum_{i<j<k<l} \gamma_i\gamma_j\gamma_k\gamma_l$

■ Nontrivial identity

$$H_4 = \frac{1}{2}\left\{(H_2)^2 - \frac{N(N-1)}{2}\right\}$$

- Proof by exhaustion

$$(H_2)^2 = -\sum_{i<j}\sum_{k<l} \gamma_i\gamma_j\gamma_k\gamma_l = \cdots$$

- Obviously, $[H_4, H_2] = 0$

■ Eigenstates of H_4

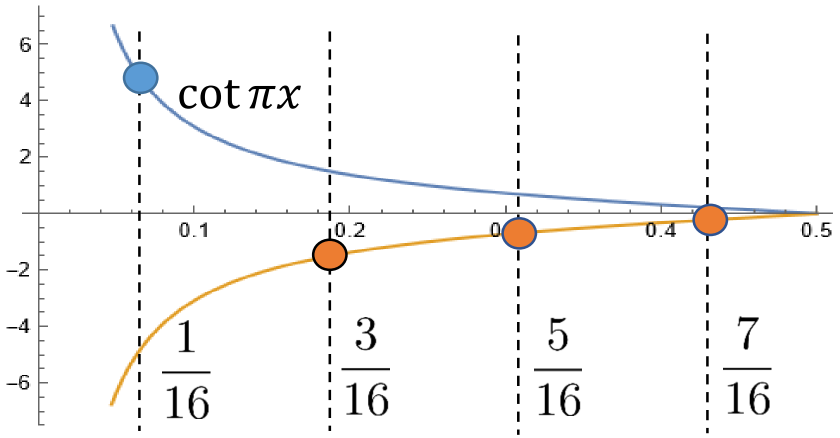
- Any eigenstate of H_2 is an eigenstate of H_4
- Solvable structure is similar to that of the Hubbard + all-to-all interaction
  Hatsugai & Kohmoto, JPSJ **61**, 2056 (1992)

1	$i<j<k<l$	$\cancel{\gamma_i\gamma_j\gamma_k\gamma_l}$
2	$i<j=k<l$	$\cancel{\gamma_i\gamma_l}$
3	$i<k<j<l$	$-\cancel{\gamma_i\gamma_k\gamma_j\gamma_l}$
4	$i<k<j=l$	$\cancel{-\gamma_i\gamma_k}$
5	$i<k<l<j$	$\cancel{\gamma_i\gamma_k\gamma_l\gamma_j}$
6	$i=k<j<l$	$\cancel{-\gamma_j\gamma_l}$
7	$i=k<j=l$	-1
8	$i=k<l<j$	$\cancel{\gamma_l\gamma_j}$
9	$k<i<j<l$	$\gamma_k\gamma_i\gamma_j\gamma_l$
10	$k<i<j=l$	$\cancel{\gamma_k\gamma_i}$
11	$k<i<l<j$	$-\cancel{\gamma_k\gamma_i\gamma_l\gamma_j}$
12	$k<i=l<j$	$\cancel{-\gamma_k\gamma_j}$
13	$k<l<i<j$	$\gamma_k\gamma_l\gamma_i\gamma_j$

$\binom{N}{2}$ cases

Ground states of H_4

- Every Fock state $|n_1, n_2, \ldots, n_{N/2}\rangle$ is an eigenstate of H_4
- Ground-states



N=8 $\underline{|1,0,0,0\rangle}$, $|0,1,1,1\rangle$

N=10 $|1,0,0,0,0\rangle$, $|0,1,1,1,1\rangle$

N=12 $|1,0,0,0,0,0\rangle$, $|0,1,1,1,1,1\rangle$

⋮

N=24 $|1,0,0,0,0,0,0,0,0,0,0,0\rangle$, $|0,1,1,1,1,1,1,1,1,1,1,1\rangle$

N=26 $|1,0,0,0,0,0,0,0,0,0,1,0,0\rangle$, $|0,1,1,1,1,1,1,1,1,1,0,1,1\rangle$

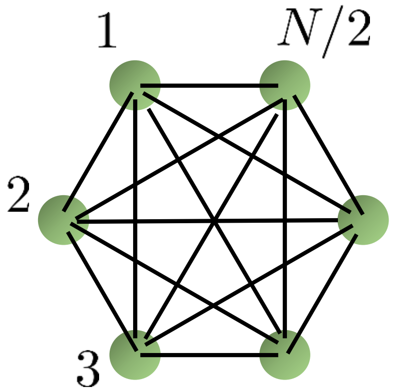
N=28 $|1,0,0,0,0,0,0,0,0,0,1,1,0,0\rangle$, $|0,1,1,1,1,1,1,1,1,1,0,0,1,1\rangle$

N=30 $|1,0,0,0,0,0,0,0,0,1,0,0,1,1,0\rangle$, $|0,1,1,1,1,1,1,1,1,0,1,1,0,0,1\rangle$

N=32 $|1,0,0,0,0,0,0,0,0,1,0,1,0,1,0,0\rangle$, $|0,1,1,1,1,1,1,1,1,0,1,0,1,0,1,1\rangle$

Hard to predict for $N \geq 26$

Equivalent Ising model



■ Spin Hamiltonian

- Occupation number $n_k = f_k^\dagger f_k \quad (k = 1, \ldots, N/2)$
- Ising variables $\sigma_k = 2n_k - 1$

$$H_4 = \sum_{k,\ell=1}^{N/2} J_{k\ell}\,\sigma_k \sigma_\ell + \text{const.} \qquad J_{k\ell} = \frac{1}{2}\cot\left(\frac{2k-1}{2N}\pi\right)\cot\left(\frac{2\ell-1}{2N}\pi\right)$$

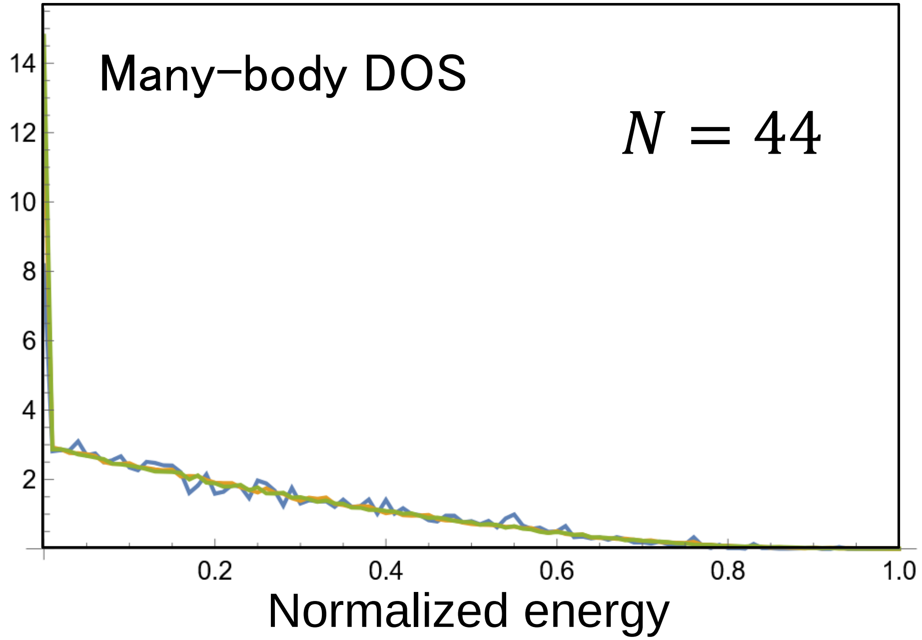
- Classical Ising model! Any Ising spin σ_k is a conserved quantity.
- Long-ranged & frustrated! glassy…?

■ Low-energy states

- Many low-energy states near the g.s.
- Number of states with $E \in [E_{\rm GS}, E_{\rm GS} + \Delta E]$

$$W(\Delta E) \simeq a \times 2^{N/2}\sqrt{\Delta E}, \quad a \simeq 0.007$$

- Entropy in the T=0 limit: $S/N \sim \frac{1}{2}\log 2$



Infinite family of models

- Clean SYK Hamiltonians (p=1,2,…)

$$H_{2p} = \mathrm{i}^p \sum_{1\le i_1<i_2<\cdots<i_{2p}\le N} \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_{2p}}$$

  - They mutually commute! $[H_{2p}, H_{2q}] = 0$ for all $p, q \in \mathbb{N}$

- Clean SYK supercharges

$$Q_{2p+1} = \mathrm{i}^p \sum_{1\le i_1<i_2<\cdots<i_{2p+1}\le N} \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_{2p+1}}$$

  - They all commute! $[Q_{2p+1}, Q_{2q+1}] = 0$
  - SUSY Hamiltonian $H_{2p}^{\mathrm{SUSY}} := (Q_{2p+1})^2$

Boost-like relation

$$Q_{2p+1} = \frac{1}{2}\{Q_1, H_{2p}\}$$

$$(Q_1 = \sum_i \gamma_i)$$

Disordered version of Q_3: Fu, Gaiotto, Maldacena & Sachdev, *PRD* **95** (2017)

- Can we prove these properties? YES!

Majorana Yang-Baxter

Sinha, Justin, Padmanabhan & Korepin,
J. Stat. Mech. 2025, 103102 (2025)

- Auxiliary Majorana ops. γ_a, γ_b
- R-matrix

$$R_{a,j}(x) = \gamma_a - x\gamma_j \quad (x \in \mathbb{C})$$

- Yang-Baxter eq.

$$R_{a,b}(x/y)R_{a,j}(x)R_{b,j}(y) = R_{b,j}(y)R_{a,j}(x)R_{a,b}(x/y)$$

  - Multiplicative form
  - Non-braided and non-local
  - Easy to check!

- Local inversion relation

$$R_{a,j}(x)^2 = 1 + x^2$$

RTT relation

- **Monodromy matrices**
  - Forward $$\overrightarrow{T}_a(u) = \prod_{j=1}^{N/2} R_{a,2j-1}(-\sqrt{\mathrm{i}u}) R_{a,2j}(\sqrt{\mathrm{i}u})$$
  - Backward $$\overleftarrow{T}_a(u) = \prod_{j=N/2}^{1} R_{a,2j}(\sqrt{\mathrm{i}u}) R_{a,2j-1}(-\sqrt{\mathrm{i}u})$$
- **RTT relation**

$$R_{a,b}(\sqrt{u/v})\overrightarrow{T}_a(u)\overrightarrow{T}_b(v) = \overrightarrow{T}_b(v)\overrightarrow{T}_a(u)R_{a,b}(\sqrt{u/v})$$

  - Same holds for the backward monodromy matrix
  - Can be proved using Majorana Yang-Baxter repeatedly
- **Global inversion relation**

$$\overrightarrow{T}_a(u)\overleftarrow{T}_a(u) = \overleftarrow{T}_a(v)\overrightarrow{T}_a(u) = (1+\mathrm{i}u)^N$$

Transfer matrices

- Decomposition of $\overrightarrow{T}_a(u)$

$$\overrightarrow{T}_a(u) = \tau_+(u) - \gamma_a \sqrt{\mathrm{i}u}\,\tau_-(u)$$

  - $\tau_\pm(u)$ commutes (anti-commutes) with fermion parity $(-1)^F = \mathrm{i}^{N/2}\gamma_1 \cdots \gamma_N$
  - They generate SYK Hamiltonians and supercharges

$$\tau_+(u) = \sum_{p=0}^{N/2} (-u)^p H_{2p}, \qquad \tau_-(u) = \sum_{p=0}^{\frac{N}{2}-1} (-u)^p Q_{2p+1}$$

  - Log derivative of $\tau_\pm(u)$ yields critical Ising/Kitaev chains

- Commuting charges
  - RTT relation implies $[\tau_+(u), \tau_+(v)] = 0, \quad [\tau_-(u), \tau_-(v)] = 0$
  - This proves the mutual commutativity!

$$[H_{2p}, H_{2q}] = 0, \quad [Q_{2p+1}, Q_{2q+1}] = 0$$

Diagonalization of transfer matrices

Diagonal forms

$$\tau_+(u) = \prod_{k=1}^{N/2} \left\{ 1 - \frac{2u}{u_k} \left(f_k^\dagger f_k - \frac{1}{2} \right) \right\}, \qquad u_k = \tan \frac{(2k-1)\pi}{2N}, \qquad \left(\frac{2}{u_k} = \epsilon_k \right)$$

$$\tau_-(u) = \sqrt{N}\chi_0 \prod_{k=1}^{\frac{N}{2}-1} \left\{ 1 - \frac{2u}{v_k} \left(g_k^\dagger g_k - \frac{1}{2} \right) \right\}, \quad \chi_0 = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \gamma_j, \quad v_k = \tan \frac{k\pi}{N}$$

- f_k (g_k): Fourier modes under APBC (PBC)
- Eigenvalues of H_{2p} and Q_{2p+1} can be read off

Key identity

- Fourier mode is invariant under conjugation with $\tau_+(u)$ up to scalar multiplication

$$\tau_+(u) f_k \tau_+(-u) = P_N^+(u^2) \frac{u_k + u}{u_k - u} f_k \qquad P_N^+(u^2) := \frac{(1+\mathrm{i}u)^N + (1-\mathrm{i}u)^N}{2}$$

- Similar relation for $\tau_-(u)$ and g_k

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Free energy and entropy

- **Reminder** $H_4 = \frac{1}{2}\left\{(H_2)^2 - \frac{N(N-1)}{2}\right\}, \quad H_2 = \sum_{k=1}^{N/2} \epsilon_k \left(f_k^\dagger f_k - \frac{1}{2}\right)$

- **Partition function**

$$Z(\beta) = \mathrm{Tr}\,\exp\left[-\beta\frac{(H_2)^2}{2}\right] = 2^{N/2}\sqrt{\frac{2}{\pi\beta}}\int_{-\infty}^{\infty} e^{-2x^2/\beta} \underline{\prod_{k=1}^{N/2}\cos(\epsilon_k x)}\, dx$$

$\sim e^{-Nx}$ for large N

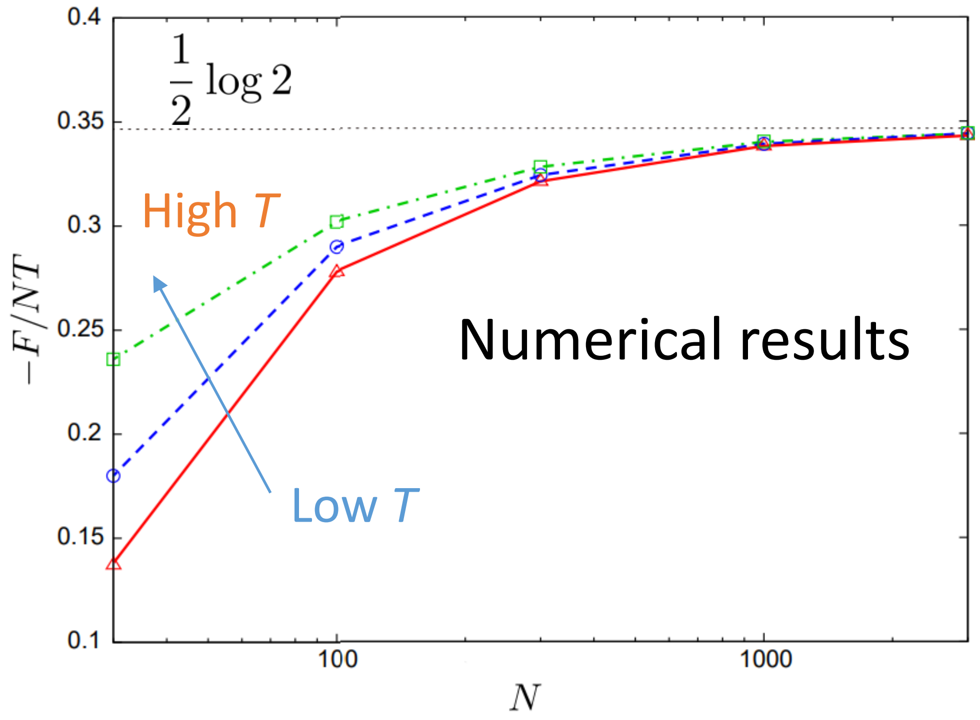
Stratonovich-Hubbard tr.

- **Free energy** $F(T) \sim -\frac{NT}{2}\log 2$

- **Entropy/N** $s(T) = -\frac{1}{N}\frac{dF}{dT} \sim \boxed{\frac{1}{2}\log 2}$
  - Original SYK

    $s(T \to 0) = \frac{1}{2}\log(1.592)$

    Maldacena et al., JHEP **106** (2016)
  - T independence
    Similar to the random energy model
    B. Derrida, Phys. Rev. B **24**, 2613 (1981)



Spectral form factor

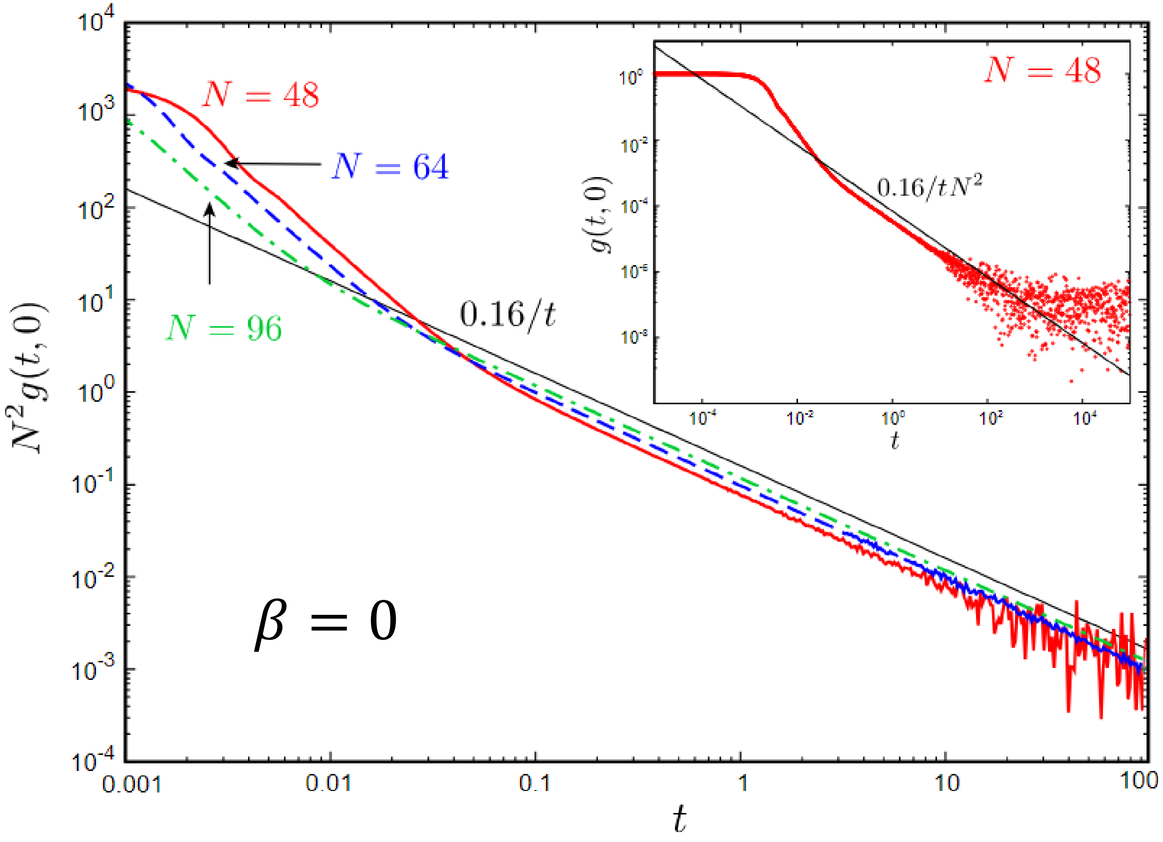
- **Definition**

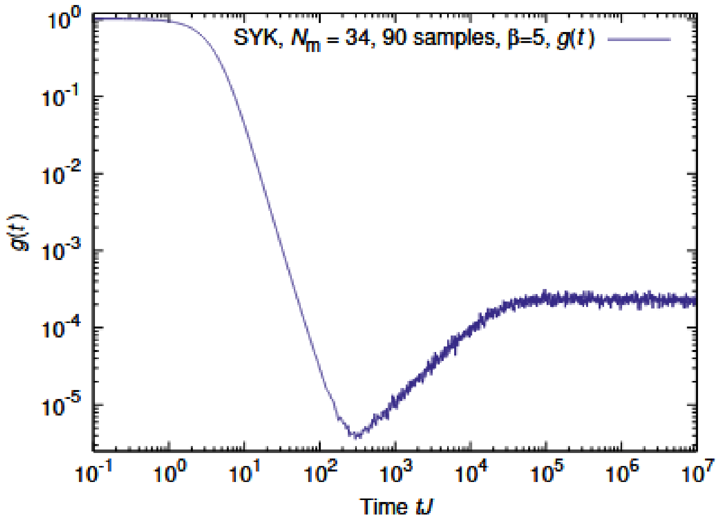
$$g(t,\beta) = \left| \frac{\mathrm{Tr} e^{(\mathrm{i}t-\beta)H_4}}{\mathrm{Tr} e^{-\beta H_4}} \right|^2$$

- **Early-time behavior**
  - Boils down to combinatorics

$$g(t,0) = 1 - \binom{N}{4} t^2 + O(t^4)$$

- **Late-time behavior**
  - Yet to get a closed form
  - But likely to be $g(t,0) \simeq \dfrac{0.16}{N^2 t}$
- **Intermediate-time region**
  - Unlike the original SYK, no dip-ramp-plateau structure





Cotler *et al.*, JHEP **5** (2017)

Time evolution by H_4

■ Warm-up: quadratic case

$$H_2 = \mathrm{i}\sum_{i<j}\gamma_i\gamma_j = \sum_{k=1}^{N/2}\epsilon_k\left(f_k^+ f_k^- - \frac{1}{2}\right), \quad f_k^+ := f_k^\dagger,\ f_k^- := f_k,\ \epsilon_k = 2\cot\frac{\theta_k}{2}$$

$$\theta_k = \frac{2k-1}{N}\pi$$

$$[H_2, f_k^\pm] = \pm\epsilon_k f_k^\pm \quad \Rightarrow \quad e^{\mathrm{i}H_2 t} f_k^\pm e^{-\mathrm{i}H_2 t} = \exp\left(\pm\mathrm{i}\epsilon_k t\right) f_k^\pm$$

- Quartic case

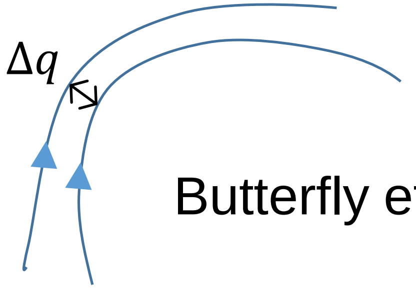
$$H_4 = -\sum_{i<j<k<l}\gamma_i\gamma_j\gamma_k\gamma_l = \frac{1}{2}\left\{(H_2)^2 - \frac{N(N-1)}{2}\right\}$$

$$[H_4, f_k^\pm] = \left(\mp\epsilon_k H_2 - \frac{1}{2}\epsilon_k^2\right) f_k^\pm \quad \Rightarrow \quad e^{\mathrm{i}H_4 t} f_k^\pm e^{-\mathrm{i}H_4 t} = \exp\left(\mp\mathrm{i}\epsilon_k \underline{H_2} t - \frac{1}{2}\epsilon_k^2 t\right) f_k^\pm$$

  - Quasi-particle picture is broken!
  - Time evolution of Majorana op.

$$\gamma_j(t) = e^{\mathrm{i}H_4 t}\gamma_j e^{-\mathrm{i}H_4 t} = \sqrt{\frac{2}{N}}\sum_{s=\pm}\sum_{k=1}^{N/2}\exp\left(\mathrm{i}s(j-1)\theta_k + \mathrm{i}s\epsilon_k H_2 t - \frac{\mathrm{i}}{2}\epsilon_k^2 t\right) f_k^s$$

Out-of-time order correlator (OTOC)



Butterfly effect

■ OTOC

- Indicator of initial value sensitivity (scrambling)
- Definition

$$C_{VW}(t) := \mathrm{tr}[\rho^{1/4}V(t)\rho^{1/4}W(0)\rho^{1/4}V(t)\rho^{1/4}W(0)]$$

Operators: V, W

Density matrix: $\rho = e^{-\beta H}/Z$

■ Quantum Lyapnov exponent λ

- Early-time behavior

$$C_{VW}(t) \sim A + Be^{\lambda t}$$

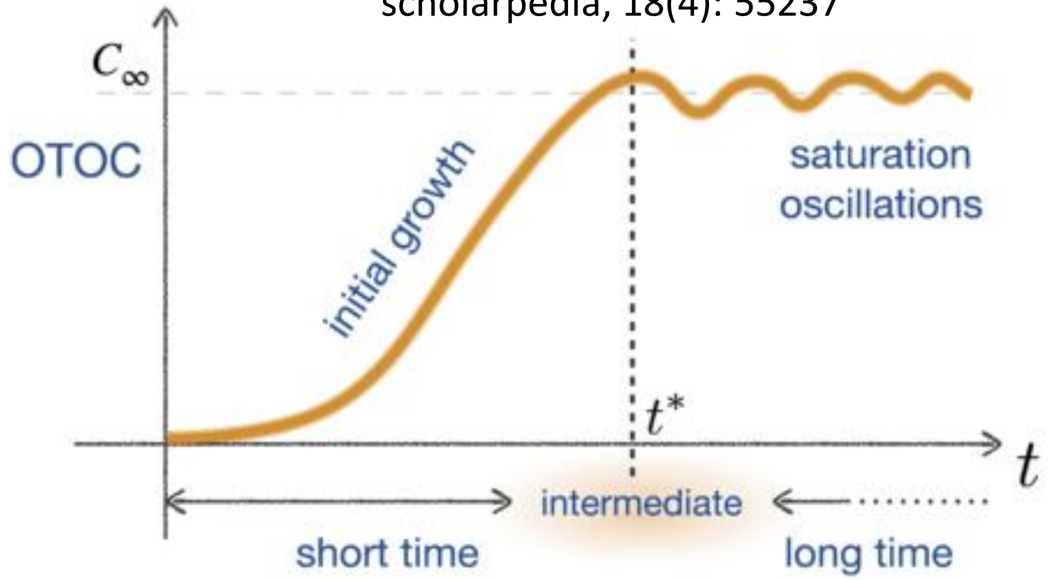
- MSS bound

$$\lambda \leq \frac{2\pi}{\beta} = 2\pi T$$

Maldacena, Shenker & Stanford, JHEP (2016)

- Original SYK saturates this bound
  - ➢ Maximally chaotic!

Garcia-Mata et al., scholarpedia, 18(4): 55237



OTOC in clean SYK H_4 at infinite T

■ OTOC of Majorana operators

$$C_{ij}(t) = \mathrm{Tr}[\rho^{1/4}\gamma_i(t)\rho^{1/4}\gamma_j(0)\rho^{1/4}\gamma_i(t)\rho^{1/4}\gamma_j(0)] \qquad \gamma_j(t) = e^{\mathrm{i}H_4 t}\gamma_j e^{-\mathrm{i}H_4 t}$$

$$F(t,\beta) = \frac{1}{N^2}\sum_{i,j=1}^{N} C_{ij}(t)$$

■ Infinite-T OTOC

$$C_{ij}(t) = \mathrm{Tr}[\gamma_i(t)\gamma_j(0)\gamma_i(t)\gamma_j(0)]$$

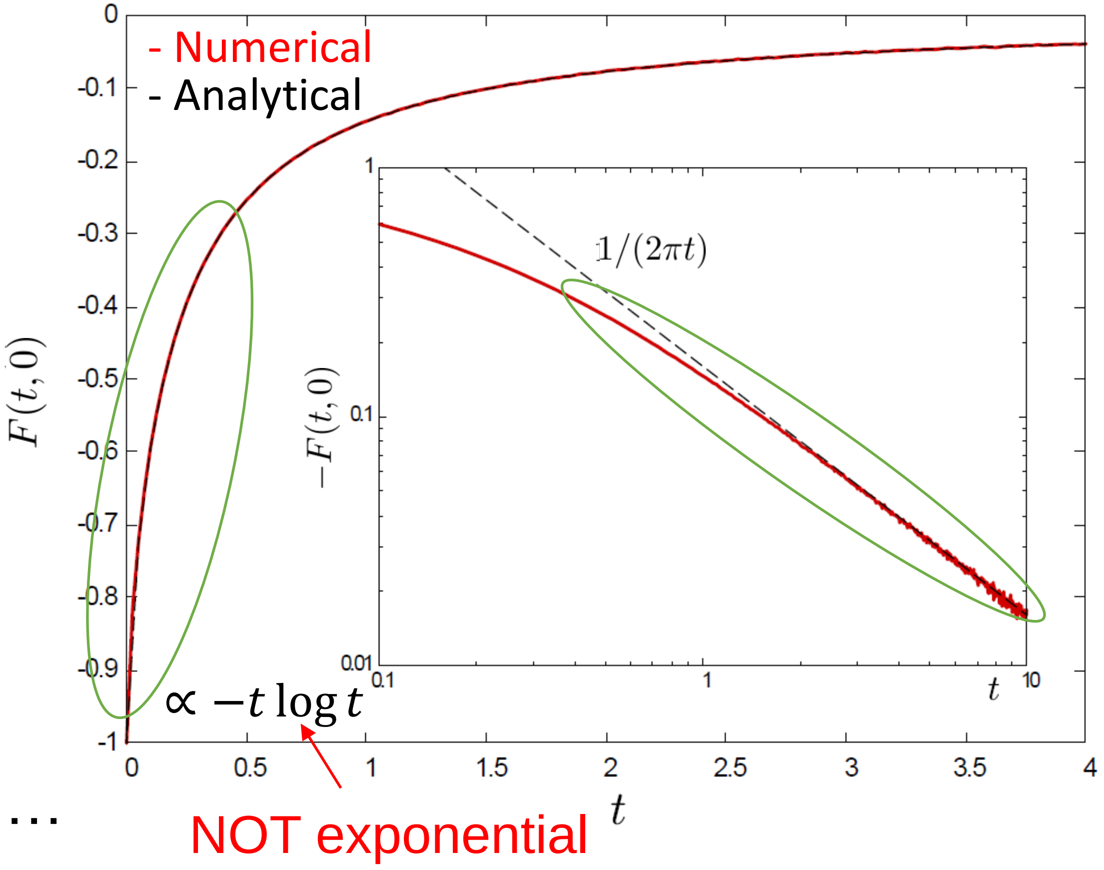
- Analytical result

$$F(t,0) = -\frac{2}{\pi}\int_0^\infty \frac{\sin u}{\sin u + 4t}du + O\left(\frac{1}{N}\right)$$

$$\sim \begin{cases} -1 + \frac{8}{\pi}(1-\gamma-\log 4t)t & (t \ll 1) \\ -\frac{1}{2\pi t} & (t \gg 1) \end{cases}$$

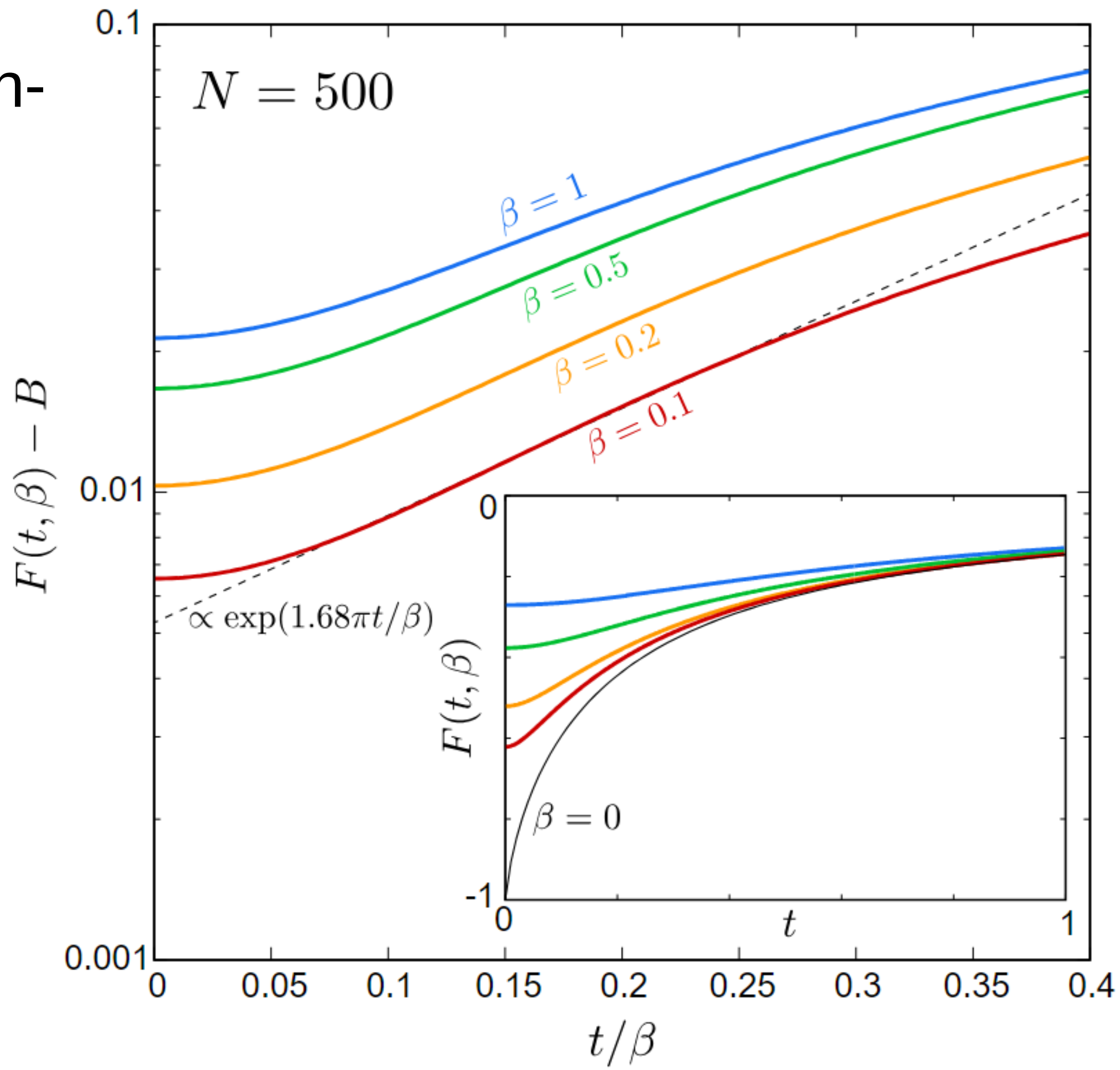
- Similar late-time behavior in integrable models: Lin & Motrunich, PRB **97** (2018), …

N=1000 particles



OTOC in clean SYK H_4 at finite T

- **Finite-T OTOC**
  - Can be computed using Stratonovich-Hubbard tr. + numerical integration
- **Early-time behavior**
  - Exponential growth at early times
  - Fitting $A + Be^{\lambda t}$ to the data in $[0.1\beta, 0.2\beta]$ yields $\lambda \simeq 1.68\pi T$
  - Consistent with the MSS bound $\lambda \leq 2\pi T$
  - Catch: much shorter time scale than that of the original SYK $t_c \sim \beta \log N$
- **What about H_2?**
  - No signature of scrambling



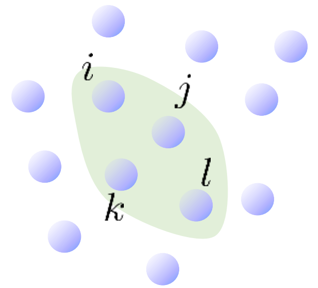
Outline

1. Majorana fermion models
   - Introduction & Motivation
   - Clean Majorana SYK
   - Clean supersymmetric Majorana SYK
2. Static and dynamical properties of H_4
   - Finite-T entropy & spectral form factor
   - Out-of-time order correlator (OTOC)
3. Complex fermion models
   - Clean complex SYK
   - Clean supersymmetric complex SYK
4. Summary

$$H_{\mathrm{c}4} = \sum_{j<i} \sum_{k<l} c_i^\dagger c_j^\dagger c_k c_l$$

$$Q_{\mathrm{c}3} = \sum_{i<j<k} c_i c_j c_k$$

Clean complex SYK model



■ Complex (spinless) fermions

- Creation and annihilation operators: $c_i^\dagger, c_j \quad (i, j = 1, 2, ..., N)$
- Canonical anti-commutation relations

$$\{c_i, c_j^\dagger\} = c_i c_j^\dagger + c_j^\dagger c_i = \delta_{i,j}, \quad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$$

■ Original model (disordered)

- Hamiltonian Sachdev, PRX **5**, (2015); Fu & Sachdev, PRB 94 (2016)

$$H_{\mathrm{cSYK}} = \sum_{1 \le j < i \le N} \sum_{1 \le k < l \le N} J_{ij;kl} c_i^\dagger c_j^\dagger c_k c_l$$

Complex Gaussian variables

$$J_{ij;kl} = -J_{ji;kl} = -J_{ij;lk} = J_{lk;ji}^*$$

$$\langle J_{ij;kl} \rangle = 0, \quad \langle |J_{ij;kl}|^2 \rangle = J^2/N^3$$

■ Clean complex SYK model

- Hamiltonian Iyoda & Sagawa, PRA **97** (2018)

$$H_{\mathrm{c}4} = \sum_{1 \le j < i \le N} \sum_{1 \le k < l \le N} c_i^\dagger c_j^\dagger c_k c_l$$

Is it Integrable?

YES! But not free-fermionic…

Integrability of H_{c4}

Iyoda, Katsura & Sagawa, PRD **98** (2018)

■ Factorization

$$\boldsymbol{c} = (c_1, \ldots, c_N)$$

$$H_{c4} = A^\dagger A \geq 0, \qquad A = \sum_{1\leq k<l\leq N} c_k c_l = \frac{1}{2}\boldsymbol{c}^{\mathrm{T}} \mathcal{A}\, \boldsymbol{c}$$

Antisymmetric matrix

$$\mathcal{A}_{ij} = \begin{cases} 1 & i<j \\ 0 & i=j \\ -1 & i>j \end{cases}$$

■ Canonical form

- $\mathcal{A}$ is non-diagonalizable, but $\mathcal{A}$ can be taken to $\mathcal{K} = \mathcal{O}\mathcal{A}\mathcal{O}^{\mathrm{T}} = \begin{pmatrix} 0 & \lambda_1 & & & O & \\ -\lambda_1 & 0 & & & & \\ & & 0 & \lambda_2 & & \\ & O & -\lambda_2 & 0 & & \\ & & & & & \ddots \end{pmatrix}$, $\quad \lambda_k = \cot\left(\frac{2k-1}{2N}\pi\right)$
- A in the new basis:

$$A = \frac{1}{2}\boldsymbol{f}^{\mathrm{T}} \mathcal{K}\, \boldsymbol{f} = \sum_{k=1}^{N/2} \lambda_k\, f_{k\uparrow} f_{k\downarrow}, \qquad \boldsymbol{f} = \mathcal{O}\boldsymbol{c} = (f_{1\uparrow}, f_{1\downarrow}, f_{2\uparrow}, f_{2\downarrow}, \ldots)$$

■ Equivalent to a known model!

Richardson, JMP **6**, 1034 (1965); Gaudin's book

- Particular case of Richardson-Gaudin model
- Bethe-ansatz solvable
- *E*=0 states are in 1-to-1 correspondence with the lowest-weight states of *η*SU(2) Yang, PRL **63** (1989)

$$\eta^- = \sum_{k=1}^{N/2} f_{k\uparrow} f_{k\downarrow}$$

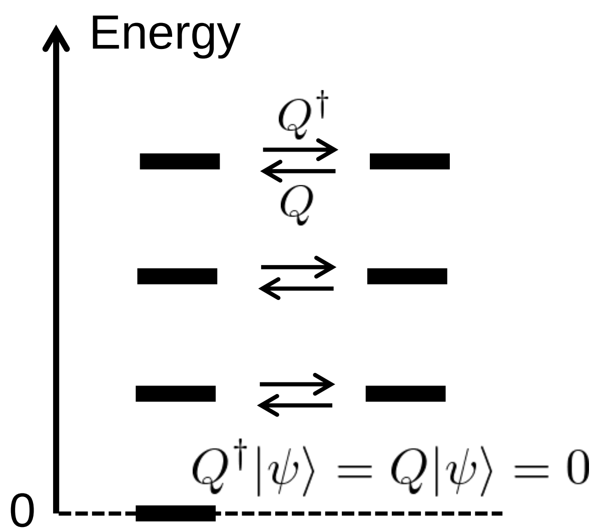
$$\dim\ker A = \dim\ker\eta^-$$

Supersymmetric version

- $\mathcal{N}=2$ SUSY quantum mechanics
  - Supercharges $Q,\ Q^\dagger, \quad Q^2=0,\ (Q^\dagger)^2=0$
  - Fermionic parity $\{Q,(-1)^F\}=\{Q^\dagger,(-1)^F\}=0$
  - Hamiltonian $H=\{Q,Q^\dagger\}=QQ^\dagger+Q^\dagger Q$
  - Symmetry $[H,Q]=[H,Q^\dagger]=[H,(-1)^F]=0$

Nicolai, JPA **9**, 1497 (1976);
Witten, NPB **202**, 253 (1982)

- Spectrum of *H*
  - $E\geq 0$ for all states
  - $E>0$ states come in pairs $\{|\psi\rangle, Q^\dagger|\psi\rangle\}$
  - $E=0$ iff a state is a SUSY singlet



- SUSY cSYK
  - Supercharge
  - Hamiltonian

$$Q_{\mathrm{cSYK}}=\mathrm{i}\sum_{1\leq i<j<k\leq N} D_{ijk}\, c_i c_j c_k$$

$$\langle D_{ijk}\rangle=0,\quad \langle |D_{ijk}|^2\rangle=\frac{2J}{N^2}$$

$$H^{\mathrm{SUSY}}_{\mathrm{cSYK}}=\{Q_{\mathrm{cSYK}},Q^\dagger_{\mathrm{cSYK}}\}$$

Fu, Gaiotto, Maldacena & Sachdev, PRD **95** (2017);
Sannomiya, Katsura & Nakayama, PRD **95** (2017)

Clean SUSY complex SYK

- Supercharge & Hamiltonian $Q_{c3} = \frac{1}{\sqrt{N}} \sum_{1\le i<j<k\le N} c_i c_j c_k$ $\quad H_{c4}^{\mathrm{SUSY}} = \{Q_{c3}, Q_{c3}^\dagger\}$
- G.S. degeneracies

H_{c4}^{SUSY}

N	3	4	5	6	7	8	9	10	11	12	13
Z_N	6	12	20	40	70	140	252	504	924	1848	3432

…

H_{c4}

N	3	4	5	6	7	8	9	10	11	12	13
Z_N	6	10	20	35	70	126	252	462	924	1716	3432

OEIS: A063886

$$Z_N = 2\binom{N}{\lfloor \frac{N}{2} \rfloor}$$

- Integrability
  - $Q_{c3} = f_0 \tilde{A}$ with $f_0 := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} c_j$ and $\tilde{A} = \sum_{j,k} (\tilde{A})_{jk} c_j c_k$

Antisymmetric circulant matrix

  - Hamiltonian

$$H_{c4}^{\mathrm{SUSY}} = \tilde{A}^\dagger \tilde{A}\, f_0^\dagger f_0 + \tilde{A}\tilde{A}^\dagger (1 - f_0^\dagger f_0)$$

Sum of two commuting Richardson-Gaudin Hamiltonians!

  - Explains the degeneracies observed numerically

Summary

■ List of integrable SYK-like models

- Clean Majorana SYK

$$H_4 = -\sum_{i<j<k<l} \gamma_i \gamma_j \gamma_k \gamma_l$$

- Clean SUSY Majorana SYK

$$Q_3 = \mathrm{i} \sum_{i<j<k} \gamma_i \gamma_j \gamma_k , \quad H_4^{\mathrm{SUSY}} = (Q_3)^2$$

- Clean complex SYK

$$H_{c4} = \sum_{j<i} \sum_{k<l} c_i^\dagger c_j^\dagger c_k c_l$$

- Clean SUSY Complex SYK

$$Q_{c3} = \sum_{i<j<k} c_i c_j c_k , \quad H_{c4}^{\mathrm{SUSY}} = \{Q_{c3}, Q_{c3}^\dagger\}$$

- Infinitely many other …

■ Discussed static and dynamical properties of H_4:

- Residual entropy reminiscent of the original SYK
- Spectral form factor: No dip-ramp-plateau structure
- OTOC: Exponential growth (early times)
  Precursor of quantum chaos?