

Applications of the modified algebraic Bethe ansatz :

Full Counting Statistics for the XXX Twisted Spin Chain
And
Rectangular 6-Vertex Model with General Boundary Conditions

[arXiv:2310.05850](https://arxiv.org/abs/2310.05850)

[arXiv:1908.00032](https://arxiv.org/abs/1908.00032)

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[arXiv:1506.06550](https://arxiv.org/abs/1506.06550)

in collaboration with **R. Pimenta, N. Slavnov, B. Vallet.**

[arXiv:2509.13928](https://arxiv.org/abs/2509.13928)

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Plan of the Talk

- Motivations
- MABA for the twisted XXX spins chain
- Modified Slavnov's formula and linear system method
- Full counting statistics for the twisted XXX spins chain (with A. Hutsalyuk)
- Rectangular 6-vertex model with general boundaries conditions (with M. Cornillault)
- Some open problems

Motivations

State of the art ABA Model with U(1) symmetry

- Usual ABA [Faddeev et al. 79]: Quantum groups, models on the circle
- Scalar product of Bethe Vectors : $Sl(2)$ [Izergin, Korepin 80's; Slavnov 89], $Sl(3)$ [B, Pakuliack, Slavnov, Ragoucy 12]
- Inverse Problem [Maillet, Kitanine, Terras 00]

Problematic for models without U(1) symmetry:

- No referent state for the transfer matrix !!!
- No decomposition of the spectrum problem !!!

- Constraints on parameters : ABA for XYZ [Faddeev et al. 79], AnBA [Nepomechie 04], ABA [Cao et al 03], CBA [Crampé et al 10],...
- Alternatives : q-Onsager [Baseilhac, Koizumi 07], Functional approach [Galleas 08]
SoV [Frahm et al. 08; Niccoli et al. 12],
ODBA [Cao et al. 13] New term for the eigenvalues and BE!!!

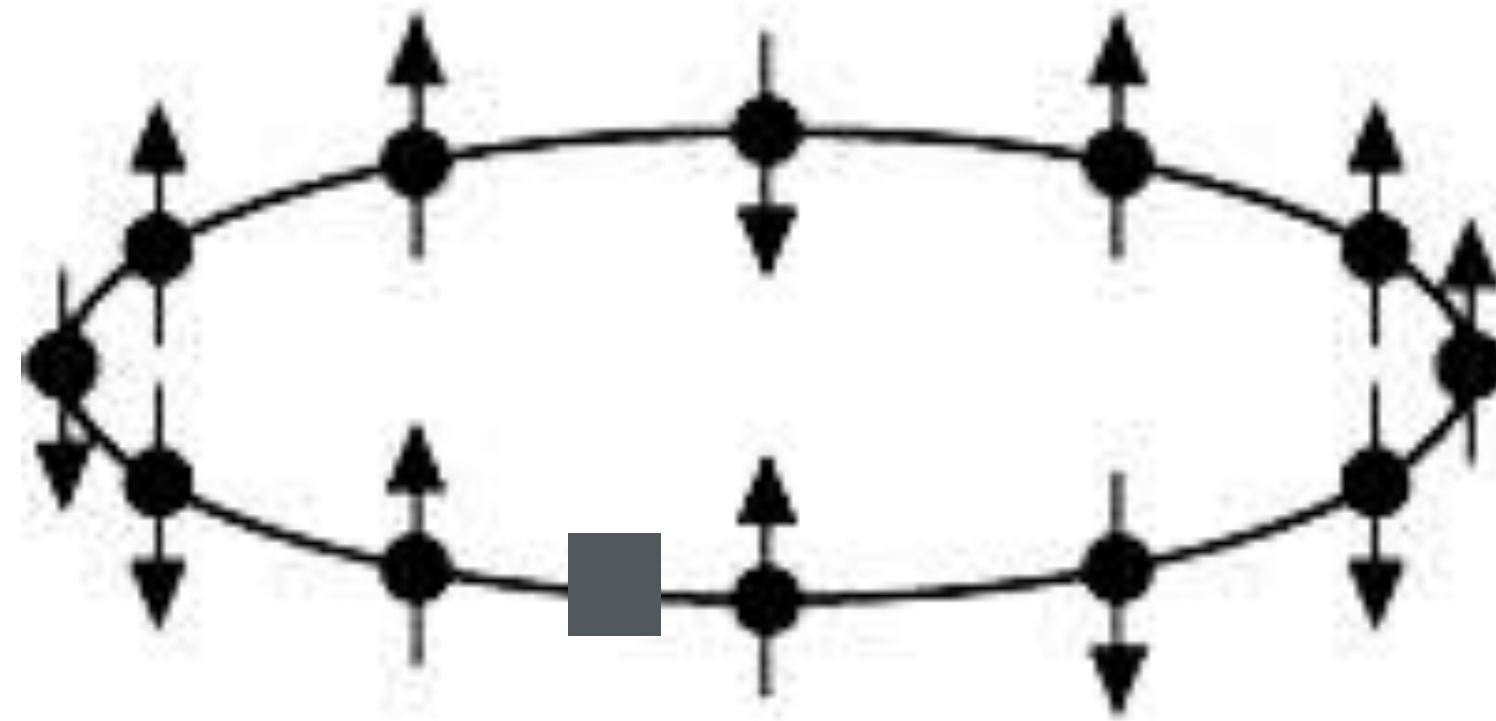
New ABA presented here : the modified ABA for models without U(1) symmetry
(sl2 case, finite Hilbert space)

[B, Crampé 13], [B 14], [Crampé 14], [B, Pimenta 15], [B, Slavnov, Vallet, 18],...

MABA for twisted XXX spins chain

Conjecture (B, Pimenta 2015)

Proof (B, Slavnov, Vallet 2018)



Twisted one dimensional spin chains

$$R(u) = u + P \quad R_{ab}(u-v)R_{ac}(u-w)R_{bc}(v-w) = R_{bc}(v-w)R_{ac}(u-w)R_{ab}(u-v).$$

R matrix

Yang-Baxter equation.

Quantum group : The Yangian

$$R_{ab}(u - v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u - v)$$

Monodromy matrix

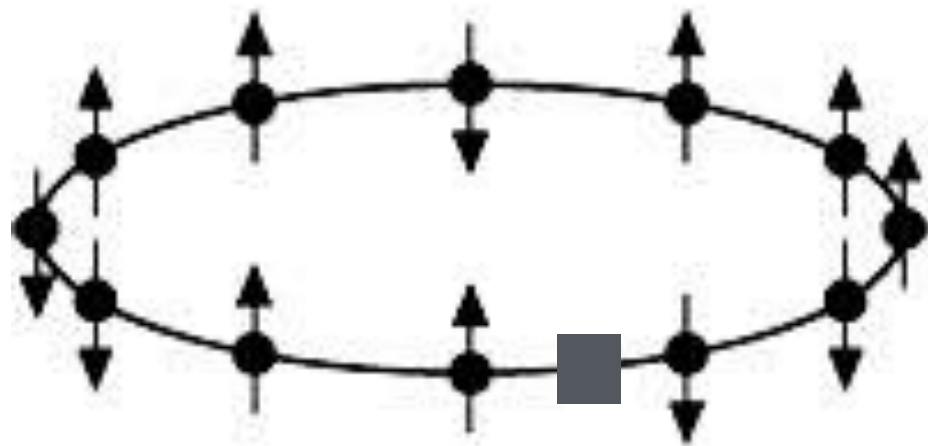
$$T_a(u) = R_{a1}(u) \dots R_{aN}(u) \qquad T(u) = \left(\begin{array}{cc} A(u) & B(u) \\ C(u) & D(u) \end{array} \right)$$

Transfer matrix

$$t(u) = tr(KT(u))$$

Twisted XXX spins chain

$$H = 2 \frac{d}{du} \left(\ln \left(t(u) \right) \right) \Big|_{u \rightarrow 0} - N$$



$$H = \sum_{k=1}^N \left(\sigma_k^x \otimes \sigma_{k+1}^x + \sigma_k^y \otimes \sigma_{k+1}^y + \sigma_k^z \otimes \sigma_{k+1}^z \right)$$

$$\sigma_{N+1}^\alpha = w_{\alpha\beta} \sigma_1^\beta$$

Modified operators

« Mysterious » Twist decomposition

$$K = LDM = \begin{pmatrix} \tilde{\kappa} & \kappa^+ \\ \kappa^- & \kappa \end{pmatrix}$$

$$D = \begin{pmatrix} \tilde{\kappa} - \rho_1 & 0 \\ 0 & \kappa - \rho_2 \end{pmatrix}$$

$$L = \sqrt{\mu} \begin{pmatrix} 1 & \frac{\rho_2}{\kappa^-} \\ \frac{\rho_1}{\kappa^+} & 1 \end{pmatrix}, \quad M = \sqrt{\mu} \begin{pmatrix} 1 & \frac{\rho_1}{\kappa^-} \\ \frac{\rho_2}{\kappa^+} & 1 \end{pmatrix}, \quad \mu = \frac{1}{1 - \frac{\rho_1 \rho_2}{\kappa^+ \kappa^-}}$$

One constraint, one free parameter

$$\kappa^+ \kappa^- - (\rho_1 \kappa + \rho_2 \tilde{\kappa}) + \rho_1 \rho_2 = 0.$$

Modified monodromy matrix

$$\bar{T}(z) = MT(z)L = \begin{pmatrix} \bar{A}(z) & \bar{B}(z) \\ \bar{C}(z) & \bar{D}(z) \end{pmatrix}$$

Modified transfert matrix

$$t(z) = \text{Tr}(D\bar{T}(z)) = (\tilde{\kappa} - \rho_1)\bar{A}(z) + (\kappa - \rho_2)\bar{D}(z)$$

Modified representation

$$\overline{A}(z)|0\rangle = \lambda_1(z)|0\rangle + \frac{\rho_2}{\kappa^+} \overline{B}(z)|0\rangle$$

$$\overline{D}(z)|0\rangle = \lambda_2(z)|0\rangle + \frac{\rho_1}{\kappa^+} \overline{B}(z)|0\rangle$$

$$\overline{C}(z)|0\rangle = \left(\frac{\rho_1}{\kappa^+} \lambda_1(z) + \frac{\rho_2}{\kappa^+} \lambda_2(z) \right) |0\rangle + \frac{\rho_1 \rho_2}{(\kappa^+)^2} \overline{B}(z)|0\rangle$$

Off shell Bethe vectors

$$\overline{B}(\bar{u})|0\rangle \qquad \# \bar{u} = M$$

Off shell action of the transfer matrix

$$t(z)\overline{B}(\bar{u})|0\rangle = \frac{\kappa^-}{\mu}\overline{B}(z)\overline{B}(\bar{u})|0\rangle + \Lambda_1^M(z, \bar{u})\overline{B}(\bar{u})|0\rangle \\ + \sum_{i=1}^M g(u_i, z)E_1^M(u_i, \bar{u}_i)\overline{B}(z)\overline{B}(\bar{u}_i)|0\rangle$$

$$\Lambda_1^M(z, \bar{u}) = (\tilde{\kappa} - \rho_1)\lambda_1(z)f(\bar{u}, z) + (\kappa - \rho_2)\lambda_2(z)f(z, \bar{u}), \\ E_1^M(u_i, \bar{u}_i) = (\kappa - \rho_2)\lambda_2(u_i)f(u_i, \bar{u}_i) - (\tilde{\kappa} - \rho_1)\lambda_1(u_i)f(\bar{u}_i, u_i).$$

For finite dimensional representation

$$\sum_{i=1}^N 2s_i = S$$

$$\frac{\kappa^-}{\mu}\overline{B}(z)\overline{B}(\bar{u}) = (\rho_1 + \rho_2)\left(F(z)g(z, \bar{u})\overline{B}(\bar{u}) + \sum_{i=1}^S g(u_i, z)F(u_i)g(u_i, \bar{u}_i)\overline{B}(z)\overline{B}(\bar{u}_i)\right)$$

Fixed number of Bethe roots

$$\#\bar{u} = \sum_{i=1}^N 2s_i = S$$

$$t(z)\overline{B}(\bar{u})|0\rangle = \Lambda^S(z, \bar{u})\overline{B}(\bar{u})|0\rangle + \sum_{i=1}^S g(u_i, z)E^S(u_i, \bar{u}_i)\overline{B}(z)\overline{B}(\bar{u}_i)|0\rangle$$

$$\begin{aligned}\Lambda(z, \bar{u}) &= (\tilde{\kappa} - \rho_1)\lambda_1(z)f(\bar{u}, z) + (\kappa - \rho_2)\lambda_2(z)f(z, \bar{u}) + (\rho_1 + \rho_2)F(z)g(z, \bar{u}), \\ E(u_i, \bar{u}_i) &= (\kappa - \rho_2)\lambda_2(u_i)f(u_i, \bar{u}_i) - (\tilde{\kappa} - \rho_1)\lambda_1(u_i)f(\bar{u}_i, u_i) + (\rho_1 + \rho_2)F(u_i)g(u_i, \bar{u}_i).\end{aligned}$$

Produce the inhomogeneous TQ equation (Off diagonal Bethe ansatz Cao & al, 13)

Scalar product of Bethe Vectors

$$S_K(\bar{u}; \bar{v}) = \langle 0 | \bar{C}(\bar{v}) \bar{B}(\bar{u}) | 0 \rangle$$

Modified Izergin-Korepin Formula (both vector are off-shell). [B,Slavnov,Vallet,18]

$$S_K^{n,m}(\bar{u}, \bar{v}) = \sum_{\bar{\xi} \Rightarrow \{\bar{\xi}_I, \bar{\xi}_{II}\}} (-\beta_1)^{n-l_I} (-\beta_2)^{n-l_{II}} \lambda_2(\bar{\xi}_I) \lambda_1(\bar{\xi}_{II}) f(\bar{\xi}_I, \bar{\xi}_{II}) K_{n,l_I}^{(\mu)}(\bar{u} | \bar{\xi}_I + c) \bar{K}_{n,l_{II}}^{(\mu)}(\bar{u} | \bar{\xi}_{II} - c).$$

$$\# \bar{u} = n \quad \# \bar{v} = m \quad \bar{\xi} = \{\bar{u}, \bar{v}\} \quad \# \bar{\xi}_I = l_I \quad \# \bar{\xi}_{II} = l_{II}$$

Definition Let $\bar{u} = \{u_1, \dots, u_n\}$, $\bar{v} = \{v_1, \dots, v_m\}$ and z be a complex number. Then the modified Izergin determinant $K_{n,m}^{(z)}(\bar{u} | \bar{v})$ is defined by

$$K_{n,m}^{(z)}(\bar{u} | \bar{v}) = \det_m \left(-z \delta_{jk} + \frac{f(\bar{u}, v_j) f(v_j, \bar{v}_j)}{h(v_j, v_k)} \right).$$

Alternatively the modified Izergin determinant can be presented as

$$K_{n,m}^{(z)}(\bar{u} | \bar{v}) = (1 - z)^{m-n} \det_n \left(\delta_{jk} f(u_j, \bar{v}) - z \frac{f(u_j, \bar{u}_j)}{h(u_j, u_k)} \right).$$

Slavnov formula (spin 1/2). One on-shell Bethe vector $\bar{u}$

$$S_K(\bar{u}; \bar{v}) = \langle 0 | \bar{B}(\bar{u}) | 0 \rangle \frac{\Delta(\bar{v}) \Delta'(\bar{u}) d(\bar{u})}{g(\bar{v}, \bar{u})} \det \left(c \frac{\partial \Lambda_K(v_k | \bar{u})}{\partial u_j} \right),$$

Conjecture (B, Pimenta 2015)

$$\#\bar{v} = \#\bar{u} = N$$

Proof from linear system method (Cramer's rule). (B, Slavnov 2019)

$$\sum_{i=0}^N M_{ji} X_i = 0.$$

$$X_i = S_K(\bar{u}_i, \bar{v})$$

Full counting statistic

(B, Hutsalyuk 2025)

$$\exp \left(Q^{(\ell)}(\bar{\beta}) \right) = \exp \left(\sum_{j=1}^{\ell} Q_j(\bar{\beta}) \right),$$

$$Q(\bar{\beta}) = (\beta_x \sigma^x + \beta_y \sigma^y + \beta_z \sigma^z)$$

$$\left\langle \exp \left(Q^{(\ell)}(\bar{\beta}) \right) \right\rangle_K = \frac{\langle K | \exp \left(Q^{(\ell)}(\bar{\beta}) \right) | K \rangle}{\langle K | K \rangle},$$

$$\tilde{K} = K \exp(-Q(\bar{\beta})),$$

$$(t_{\tilde{K}}(0))^{-l} (t_K(0))^l = \prod_{i=1}^l \tilde{K}_i^{-1} K_i = \prod_{i=1}^l \exp(Q_i(\bar{\beta})).$$

Inverse problem (Maillet Terras 00)

$$\left\langle \exp \left(Q^{(\ell)}(\bar{\beta}) \right) \right\rangle_K = \sum_{\tilde{K}} \left(\frac{\Lambda_K(0)}{\Lambda_{\tilde{K}}(0)} \right)^l \frac{\langle K | \tilde{K} \rangle \langle \tilde{K} | K \rangle}{\langle K | K \rangle \langle \tilde{K} | \tilde{K} \rangle}.$$

Calculation of the overlap from Modified Slavnov formula

Use free parameter of the twist decomposition

$$\bar{B}(u)/\mu = \tilde{B}(u)/\tilde{\mu},$$

$$\frac{\rho_1}{\kappa_-} = \frac{\tilde{\rho}_1}{\tilde{\kappa}_-}, \quad \frac{\rho_2}{\kappa_-} = \frac{\tilde{\rho}_2}{\tilde{\kappa}_-},$$

$$\rho_1 \rho_2 + \kappa_- \kappa_+ - (\kappa_1 \rho_2 + \kappa_2 \rho_1) = 0, \quad \tilde{\rho}_1 \tilde{\rho}_2 + \tilde{\kappa}_- \tilde{\kappa}_+ - (\tilde{\kappa}_1 \tilde{\rho}_2 + \tilde{\kappa}_2 \tilde{\rho}_1) = 0.$$

$$\langle \tilde{K}, \bar{v} | \bar{u}, K \rangle = \langle 0 | \tilde{C}(\bar{v}) \bar{B}(\bar{u}) | 0 \rangle = (\mu/\tilde{\mu})^L \langle 0 | \tilde{C}(\bar{v}) \tilde{B}(\bar{u}) | 0 \rangle = (\mu/\tilde{\mu})^L S_{\tilde{K}}(\bar{v} | \bar{u}).$$

$$\langle K, \bar{u} | \bar{v}, \tilde{K} \rangle = \langle 0 | \bar{C}(\bar{u}) \tilde{B}(\bar{v}) | 0 \rangle = (\tilde{\mu}/\mu)^L \langle 0 | \bar{C}(\bar{u}) \bar{B}(\bar{v}) | 0 \rangle = (\tilde{\mu}/\mu)^L S_K(\bar{u} | \bar{v}).$$

Close formulas functions of Bethe roots

$$\left\langle \exp \left(Q^{(\ell)}(\bar{\beta}) \right) \right\rangle_K = \sum_{\bar{v}} \frac{1}{g(\bar{u}, \bar{v})g(\bar{v}, \bar{u})} \left(\frac{\Lambda_K(0|\bar{u})}{\Lambda_{\tilde{K}}(0|\bar{v})} \right)^\ell \frac{\det_L \left(c \frac{\partial \Lambda_K(v_k|\bar{u})}{\partial u_j} \right) \det_L \left(c \frac{\partial \Lambda_{\tilde{K}}(u_k|\bar{v})}{\partial v_j} \right)}{\det_L \left(c \frac{\partial \mathcal{Y}_K(u_k|\bar{u})}{\partial u_j} \right) \det_L \left(c \frac{\partial \mathcal{Y}_{\tilde{K}}(v_k|\bar{v})}{\partial v_j} \right)},$$

$$\left\langle \exp \left(Q^{(\ell)}(\bar{\beta}) \right) \right\rangle_K = \frac{1}{L!} \oint_{\bar{v}} \frac{d\bar{z}}{(2\pi)^L} \frac{1}{g(\bar{u}, \bar{z})g(\bar{z}, \bar{u})} \left(\frac{\Lambda_K(0|\bar{u})}{\Lambda_{\tilde{K}}(0|\bar{z})} \right)^\ell \frac{\det_L \left(c \frac{\partial \Lambda_K(z_k|\bar{u})}{\partial u_j} \right) \det_L \left(c \frac{\partial \Lambda_{\tilde{K}}(u_k|\bar{z})}{\partial z_j} \right)}{\det_L \left(c \frac{\partial \mathcal{Y}_K(u_k|\bar{u})}{\partial u_j} \right) \prod_{j=1}^L \mathcal{Y}_{\tilde{K}}(z_j|\bar{z})}.$$

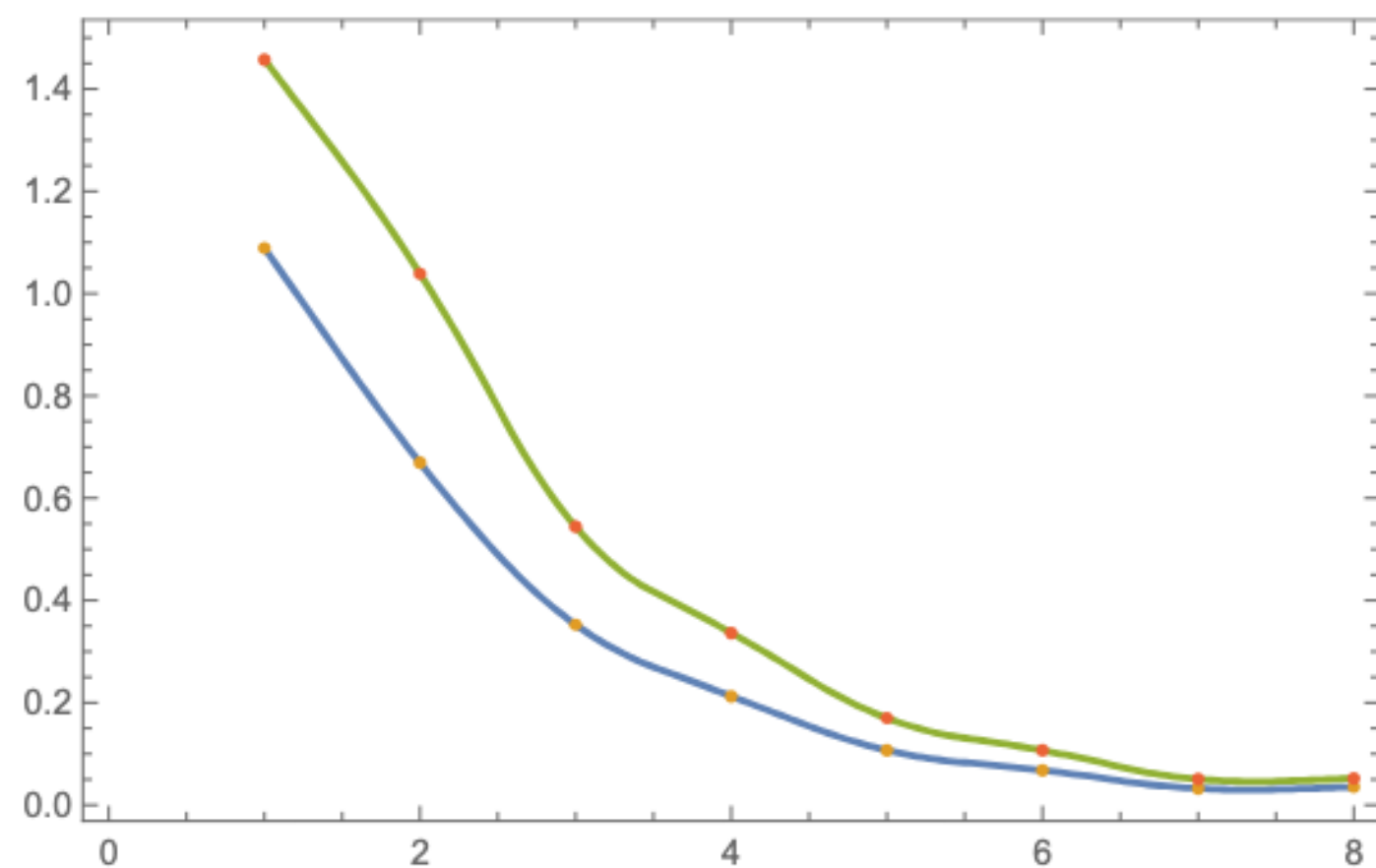


FIG. 1. FCS, $K = \sigma^x$, expectation values of $\exp(Q^{(\ell)}(\bar{\beta}))$ with $\bar{\beta} = \{1, 0, 1\}$ (the blue curve) and $\bar{\beta} = \{1, 1, 1\}$ (the green curve) are shown, $L = 8$. Dots correspond to direct diagonalization, while curves are interpolation of results derived via MABA.

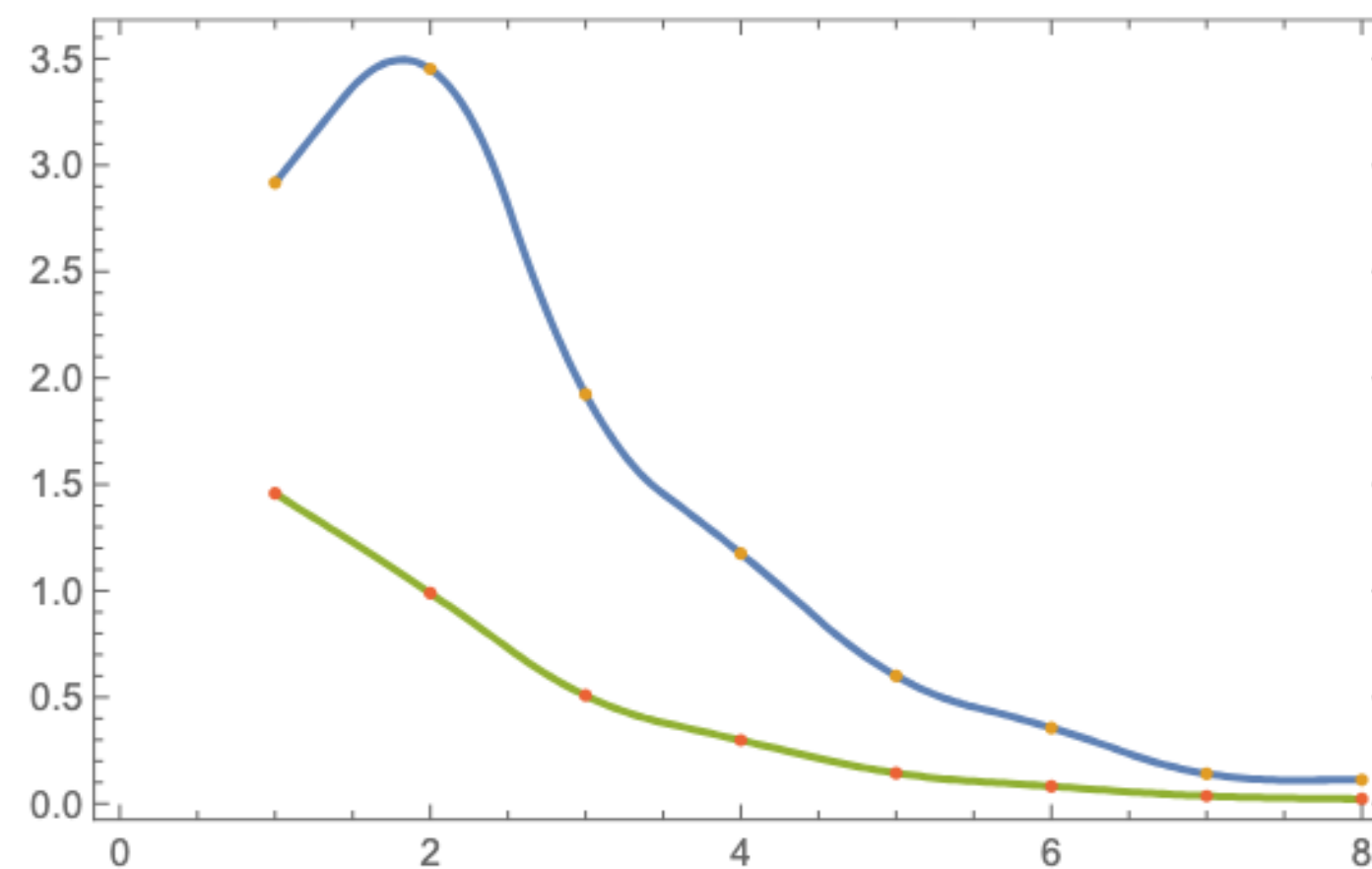


FIG. 3. FCS, $K = \sigma^x + \sigma^z$, expectation values of $\exp(Q^{(\ell)}(\bar{\beta}))$ with $\bar{\beta} = \{1, -1, 2\}$ (the blue curve) and $\bar{\beta} = \{1, 1, 1\}$ (the green curve). Dots correspond to direct diagonalization, while curves are interpolation of results derived via MABA.

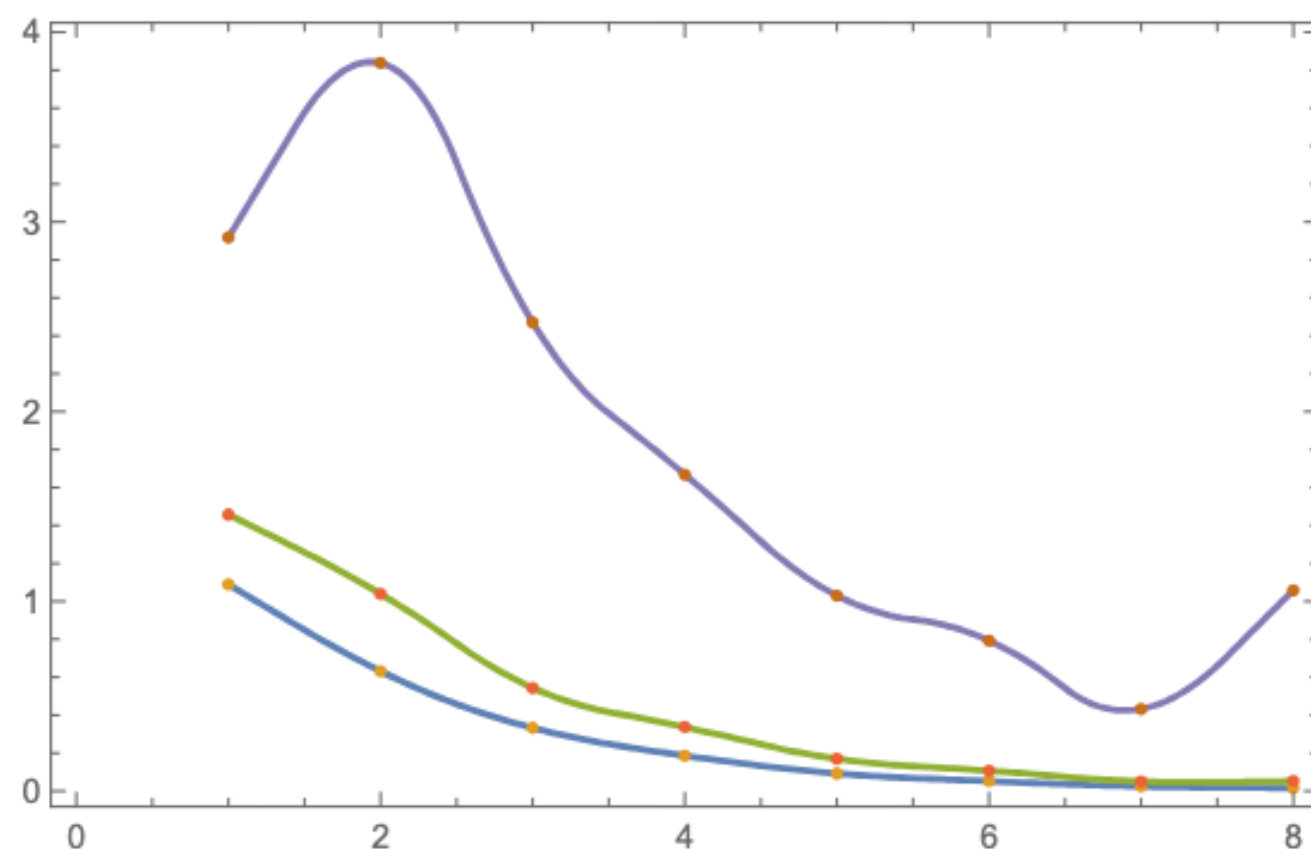
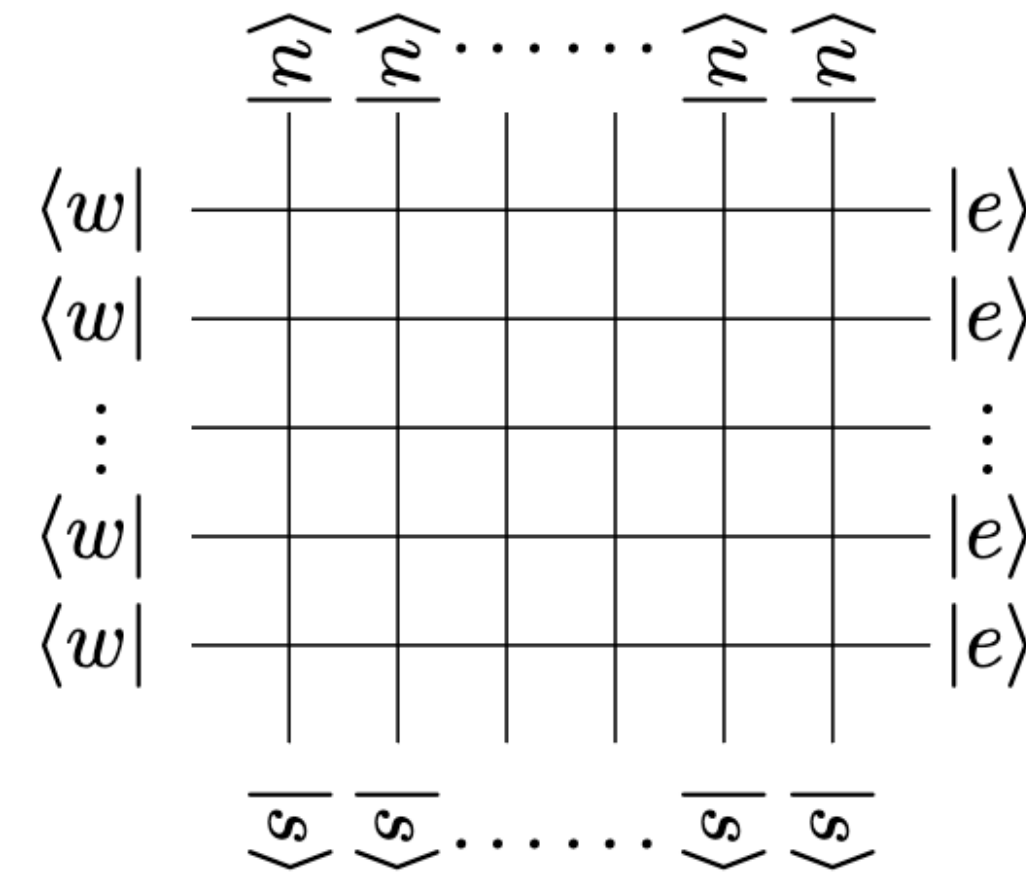
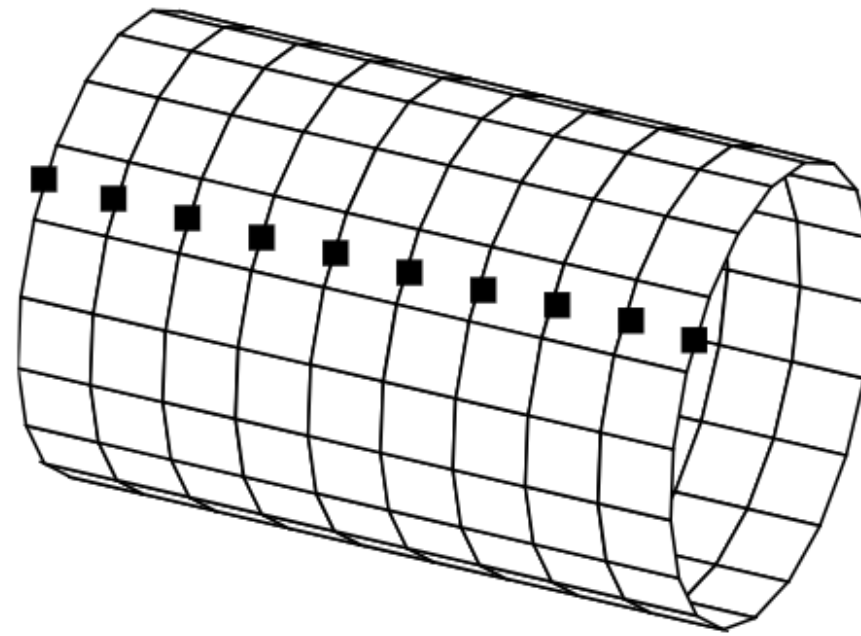
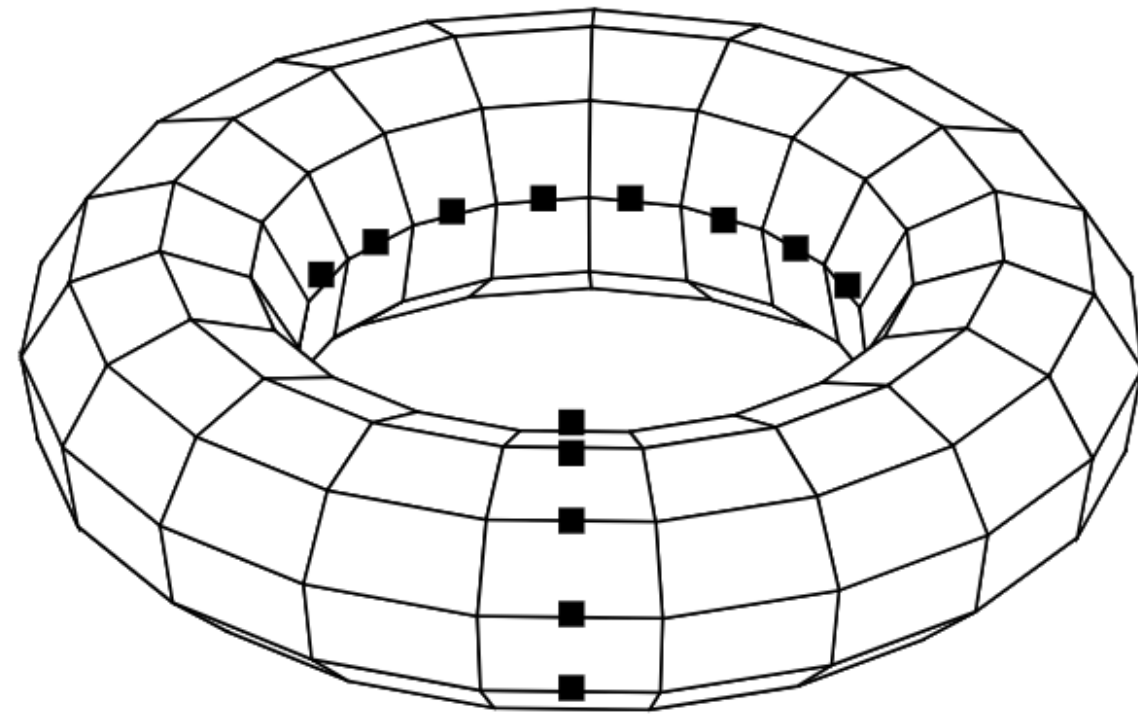


FIG. 2. FCS, $K = \sigma^y$, expectation values of $\exp(Q^{(\ell)}(\bar{\beta}))$ with $\bar{\beta} = \{1, 0, 1\}$ (the blue curve) and $\bar{\beta} = \{1, 1, 1\}$ (the green curve) and $\bar{\beta} = \{1, -1, 2\}$ (purple curve) are shown, $L = 8$. Dots correspond to direct diagonalization, while curves are interpolation of results derived via MABA.

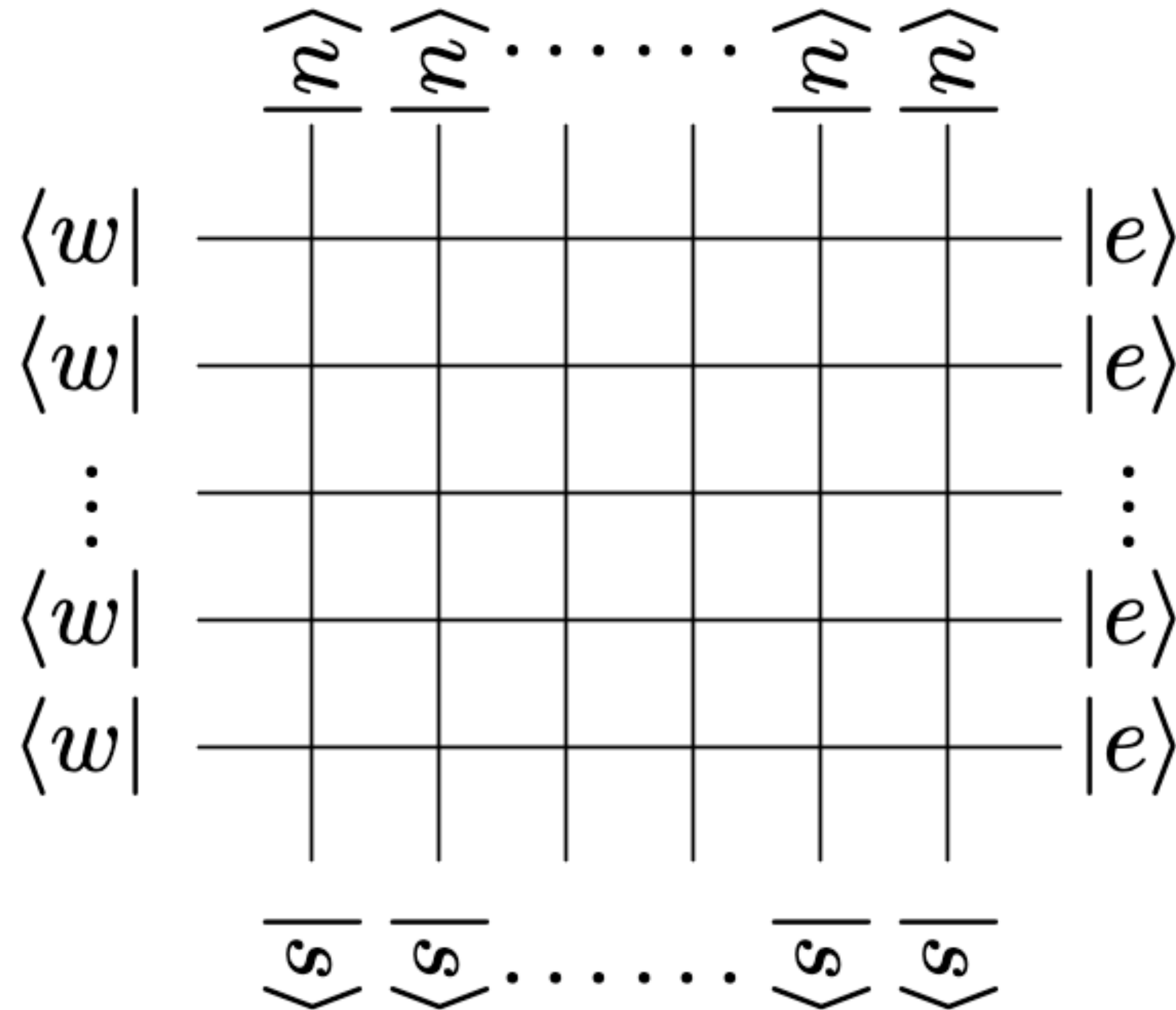
6-vertex model



Definition 1.1. *The partition function of the inhomogeneous six vertex model on the $n \times m$ rectangular lattice with general boundary conditions is*

$$Z_{nm}(\bar{u}|\bar{v}|B|\hat{C}) = \text{tr}_{\bar{a},\bar{b}} \left(\prod_{i=1}^n B_{a_i} \prod_{j=1}^m \hat{C}_{b_j} \prod_{i=1}^n \prod_{j=1}^m R_{a_i b_j}(u_i - v_j) \right). \quad (1.7)$$

Rectangular 6-vertex model with GB



(B. Pimenta, Slavnov 2024)

Definition 1.2. We define the following twist

$$B = |e\rangle \otimes \langle w| \quad \text{and} \quad \hat{C} = |s\rangle \otimes \langle n|.$$

We can rewrite the partition function with the following formula

$$Z_{nm}(\bar{u}|\bar{v}|B|\hat{C}) = \langle W| \otimes \langle N| \left(\prod_{i=1}^n \prod_{j=1}^m R_{a_i b_j}(u_i - v_j) \right) |E\rangle \otimes |S\rangle$$

with

$$\langle N| = \bigotimes_{j=1}^m \langle n|_{b_j}, \quad \langle W| = \bigotimes_{i=1}^n \langle w|_{a_i}, \quad |S\rangle = \bigotimes_{j=1}^m |s\rangle_{b_j} \quad \text{and} \quad |E\rangle = \bigotimes_{i=1}^n |e\rangle_{a_i}.$$

Proposition 1.1. *[BPS24] The partition function of the inhomogeneous rational six-vertex model with general boundary conditions is*

$$Z_{nm}(\bar{u}|\bar{v}|B|\hat{C})=\frac{\mathrm{tr}(B)^n\mathrm{tr}(\hat{C})^m}{(1-\beta)^m}\frac{1}{g(\bar{u},\bar{v})}K_{nm}^{(\beta)}(\bar{u}|\bar{v})$$

with

$$\beta=1-\frac{\mathrm{tr}(B)\mathrm{tr}(\hat{C})}{\mathrm{tr}(B\hat{C})},\quad \mathrm{tr}(B)=\langle w|e\rangle,\quad \mathrm{tr}(\hat{C})=\langle n|s\rangle\quad \text{and}\quad \mathrm{tr}(B\hat{C})=\langle n|e\rangle\langle w|s\rangle.$$

(Gorsky, Zabrodin, Zotov, 2014)

The modified Izergin determinant can be expressed with two formulae [BSV18]

$$\begin{aligned} K_{nm}^{(\beta)}(\bar{u}|\bar{v}) &= \det_{1\leqslant k,l\leqslant m}\left(-\beta\delta_{kl}+\frac{f(\bar{u},v_k)f(v_k,\bar{v}_k)}{h(v_k,v_l)}\right) \\ &= (1-\beta)^{m-n}\det_{1\leqslant k,l\leqslant n}\left(f(u_k,\bar{v})\delta_{ij}-\beta\frac{f(u_k,\bar{u}_k)}{h(u_k,u_l)}\right). \end{aligned}$$

Homogeneous limit

(Cornillault, B 2025)

(Foda, Wheeler 2012)

$$x=u-v.$$

$$\begin{aligned} Z_{nm}(x|B|\hat{C}) &= \frac{\mathrm{tr}(B)^n\mathrm{tr}(\hat{C})^m}{(1-\beta)^{\min(n,m)}} \\ &\quad \times \det_{1\leqslant i,j\leqslant \min(n,m)}\left(\binom{d+i+j-2}{i-1}\left(\left(1+\frac{x}{c}\right)^{d+i+j-1}-\beta\left(\frac{x}{c}\right)^{d+i+j-1}\right)\right). \end{aligned}$$

$$\text{with } d=|n-m|.$$

Thermodynamic limit

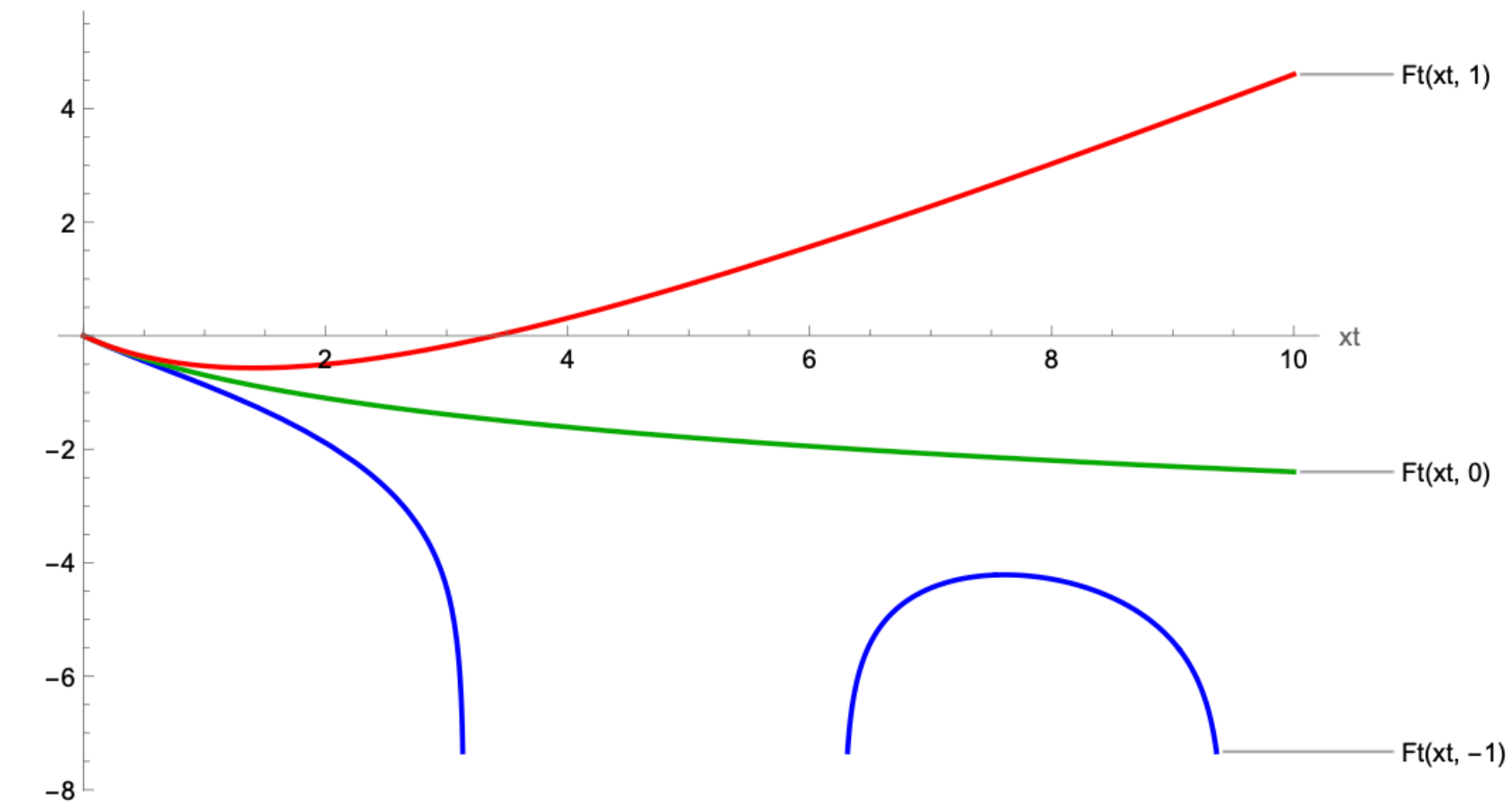
$$\ln(Z_{nm}(x|B|\hat{C})) = -nm \frac{F(x)}{k_B T} + o(nm)$$

$$\tilde{\beta} = \begin{cases} 0 & \text{if } d > 1 \\ 3\beta & \text{if } d = 1 \\ 3\beta(\beta - 2) & \text{if } d = 0 \end{cases}$$

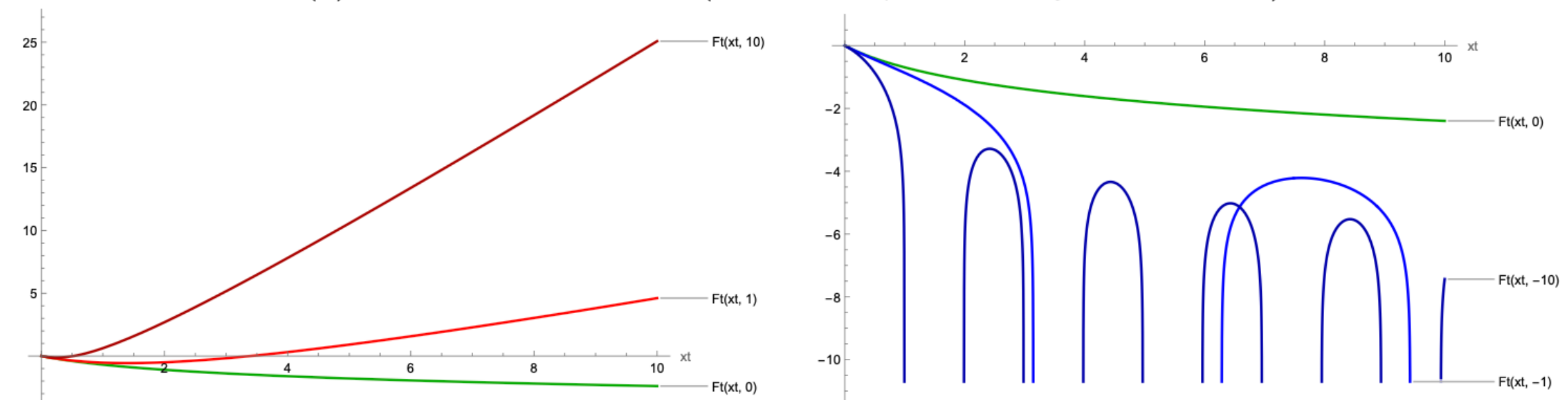
$$\tilde{F}(\tilde{x}) = \frac{F(c\tilde{x})}{k_B T} = \begin{cases} \ln \left(\frac{\sinh(\sqrt{\tilde{\beta}}\tilde{x})}{\sqrt{\tilde{\beta}}\tilde{x}} \frac{1}{1+\tilde{x}} \right) & \text{if } \tilde{\beta} > 0 \\ \ln \left(\frac{1}{1+\tilde{x}} \right) & \text{if } \tilde{\beta} = 0 \\ \ln \left(\frac{\sin(\sqrt{-\tilde{\beta}}\tilde{x})}{\sqrt{-\tilde{\beta}}\tilde{x}} \frac{1}{1+\tilde{x}} \right) & \text{if } \tilde{\beta} < 0 \end{cases}.$$

(Cornillault, B 2025)

(Korepin Zinn-Justin 00)



(a) For $\tilde{\beta} = -1, 0$ and 1 (respectively in blue, green and red).



(b) For $\tilde{\beta} = 0, 1$ and 10 (respectively in green, red and dark-red).

(c) For $\tilde{\beta} = 0, -1$ and -10 (respectively in green, blue and dark-blue).

Figure 4: Graphs of $\tilde{F}$ for different values of $\tilde{\beta}$. “xt” corresponds to $\tilde{x}$ and “Ft” to $\tilde{F}$

Some open problems

- XXZ and XYZ cases
- Thermodynamical limit for correlation function
- Open boundaries cases (FCS and Vertex Interpretation)
- Higher rank cases MABA still to be find...

Some open problems

- XXZ and XYZ cases
- Thermodynamical limit
- Open boundaries cases
- Higher rank cases
- « Et au-delà ? »