Symmetry-resolved entanglement in 1+1D conformal field theories with non-invertible symmetries

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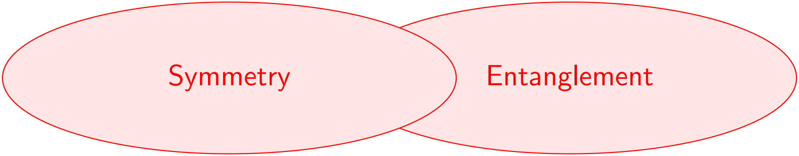
Motivation

Symmetry and (ground state) entanglement are two of the most important concepts in the description of quantum many-body systems



Order parameters

Classification of phases
($\rightarrow$ Landau's paradigm)

Selection rules for RG flows

Criticality
Topological properties

Recent years have witnessed a revolution in our understanding of symmetries, with implications across both areas.

Entanglement scaling in Conformal Field Theories (CFTs)

Question: If we know the ground state of a critical 1+1D quantum lattice model, can we determine the associated CFT?

The entanglement entropy for a subregion of size ℓ in an infinite system should scale as

$$S = \frac{c}{3} \log \ell + \cdots$$

The central charge c partly characterizes the CFT.

[Holzey,Larson,Wilczek'94] [Calabrese,Cardy'04]

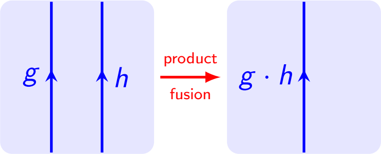
Various generalizations of this formula are known.

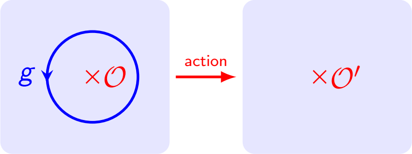
Generalized symmetries (I)

Novel standard perspective

[Fröhlich,Fuchs,Runkel,Schweigert'0x] [Gaiotto,Kapustin,Seiberg,Willett'15]

Associate symmetries with topological defects



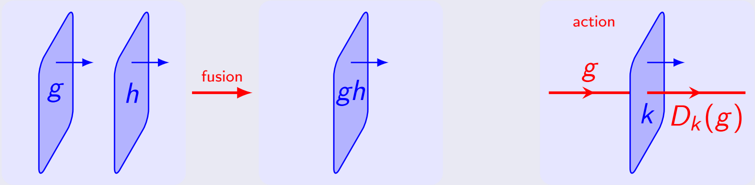


Benefits

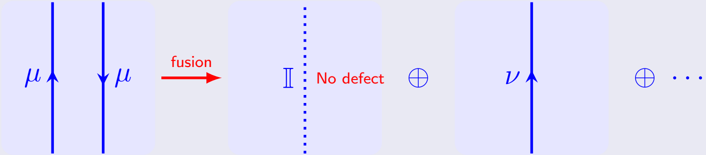
- Can realize "higher symmetries" of different dimensionality
- Can realize "non-invertible symmetries"

Generalized symmetries (II)

Higher symmetries



Non-invertible symmetries



The relevant mathematical structures are higher fusion categories

A generalized Landau paradigm

Generalized Landau paradigm

Generalized symmetries (and their breaking) can be used to

- classify quantum phases of matter (including topological ones)
- construct (local and non-local) order parameters
- establish dualities between different systems

Further benefits of generalized symmetries

- generalize the notion of anomalies
- impose selection rules for correlation functions
- constrain renormalization group flows

Combining symmetry and entanglement

The study of the interplay between symmetry and entanglement has been a rich endeavor

Classification of gapped phases

- Tensor network realization (MPS, PEPS, ...) [Hastings'06]
- Tensors reflect the symmetry [Sanz,Wolf,Perez-Garcia,Cirac'09]
- Classification of SPT phases (in 1D), etc.

[Pollmann,Turner,Berg,Oshikawa'10] [Schuch,Perez-Garcia,Cirac'11] [Duivenvoorden,Quella'12]

More recently

- Symmetry-resolved entanglement (SREE) [Goldstein,Sela'18]
- Entanglement asymmetry [Ares,Murciano,Calabrese'23]

Plan for this talk

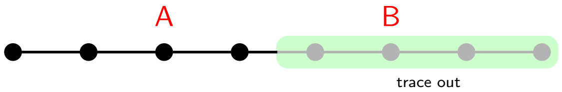
Outline of this talk

- Background information
- Examples: Generalized symmetries in CFTs
- How to describe SREE in CFTs?
- Numerical verification

Background information

Bipartite entanglement

Consider bipartition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



Ground state density matrix $\rho \to$ reduced density matrix

$$\rho_A = \mathrm{tr}_B(\rho)$$

Renyi and von Neumann entropies:

$$S_n = \frac{1}{1-n} \log \mathrm{tr}_A(\rho_A^n) \qquad \text{and} \qquad S_{EE} = -\,\mathrm{tr}_A(\rho_A \log \rho_A) = \lim_{n \to 1} S^{(n)}$$

Question: How much entanglement is there in a given symmetry sector r (an irreducible representation)?

In this case we need to insert projectors Π_r at appropriate places and revisit normalizations. The formulas become [Di Giulio,Meyer,Northe,Scheppach,Zhao'23]

$$\mathcal{Z}_n(r) = \mathrm{tr}\Big[\Pi_r \rho_A^n\Big] \qquad \text{(replica partition function)}$$

$$S_n(r) = \frac{1}{1-n} \log \frac{\mathcal{Z}_n(r)}{\mathcal{Z}_1(r)^n} \qquad S_{EE}(r) = \lim_{n \to 1} S_n(r)$$

The result is called the symmetry-resolved entanglement entropy (SREE)

Early results on symmetry-resolved entanglement in CFTs

Systems with $U(1)$ symmetry

Asymptotic equipartition of entanglement across charge sectors

[Laflorencie,Rachel'14] [Goldstein,Sela'18] [Xavier,Alcaraz,Sierra'18]

Compact Lie groups such as $SU(2)$

Equipartition is violated at subleading order by a term $\log\dim(r)$ where $\dim(r)$ is the dimension of the irreducible representation r.

[Calabrese,Dubail,Murciano '21] [Kusuki,Murciano,Ooguri,Pal'23]

See also: [Di Giulio,Meyer,Northe,Scheppach,Zhao'23] [Northe'23] [holographic approaches $\longrightarrow$ Meyer et al]

Question: What about resolution wrt generalized symmetries?

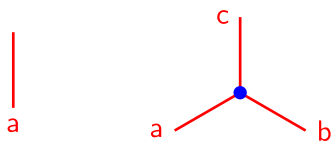
[Saura-Bastida,Das,Sierra,Molina-Vilaplana'24]
[Das,Molina-Vilaplana,Saura-Bastida'24] [Choi,Rayhaun,Zheng'24] [TQ,Heymann'25]
[Yan,Murciano,Calabrese,Konik'25]
[...more to come...]

Mathematical structure of generalized symmetries

Symmetry categories

[Bhardwaj,Tachikawa'18]

In 1+1D, finite symmetries are described by unitary fusion categories



Fusion category

- Objects: Topological line operators ($\rightarrow$ symmetries)
- Morphisms: Junction operators •
- Fusion structure: $a \otimes b = \sum_c N_{ab}^c \, c$ with $N_{ab}^c \in \mathbb{Z}_{\geq 0}$
- Associativity: $\alpha_{a,b,c} : (a \otimes b) \otimes c \rightarrow a \otimes (b \otimes c)$

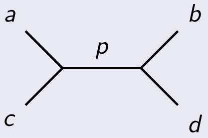
Duality and arrows

Indecomposable defect lines a have a dual $\bar{a}$ such that

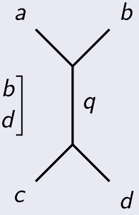
$$a \otimes \bar{a} = \mathbb{I} \oplus \cdots$$

In this talk, we will mostly consider self-dual fusion categories, so $a = \bar{a}$. This means that we can ignore line orientations.

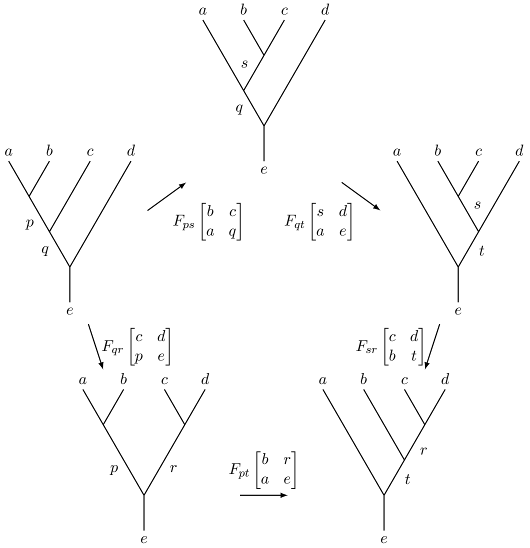
F-moves (multiplicity-free case)



$$= \sum_q F_{pq} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$



Pentagon identity (from associativity)



Quantum dimensions

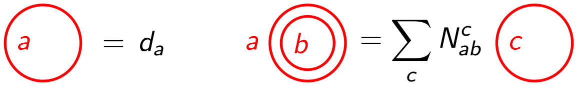
Every object a in a symmetry category carries a quantum dimension d_a.

Perron-Frobenius approach

- Define matrices $(N_a)_b^c = N_{ab}^c$
- These matrices have non-negative entries
- Define d_a to be the maximal non-negative eigenvalue of N_a
- The d_a form a 1D representation of the fusion algebra

$$d_a d_b = \sum_c N_{ab}^c d_c$$

Graphically:

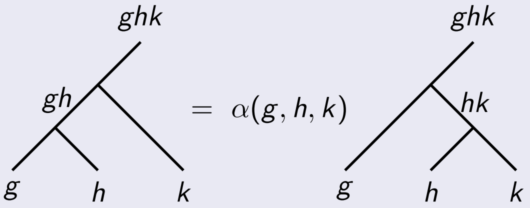


For every finite group G there exists a symmetry category Vec_G

- Simple objects: Elements g of G
- Fusion: $g \otimes h = g \cdot h$
- Dual: $\bar{g} = g^{-1}$

Inclusion of an anomaly

One can include an anomaly $\alpha \in H^3(G, U(1))$. This defines Vec_G^α



Example: The Fibonacci and Ising categories

We now explicitly adopt the perspective of defect lines: $a \to \mathcal{L}_a$

Fibonacci category

One non-trivial defect line $\mathcal{L}_\varepsilon$ satisfying

$$\mathcal{L}_\varepsilon \otimes \mathcal{L}_\varepsilon = \mathcal{L}_\mathbb{I} \oplus \mathcal{L}_\varepsilon \qquad \rightarrow \qquad \text{non-invertible} \quad d_\varepsilon = \varphi = \tfrac{1}{2}(1+\sqrt{5})$$

Ising category

Two non-trivial defect lines $\mathcal{L}_\varepsilon$ (spin flip) and $\mathcal{L}_\sigma$ (self-duality) satisfying

$$\begin{aligned}
\mathcal{L}_\varepsilon \otimes \mathcal{L}_\varepsilon &= \mathcal{L}_\mathbb{I} && \rightarrow \quad \mathbb{Z}_2 \text{ subcategory} \quad d_\varepsilon = 1 \\
\mathcal{L}_\varepsilon \otimes \mathcal{L}_\sigma &= \mathcal{L}_\sigma && \\
\mathcal{L}_\sigma \otimes \mathcal{L}_\sigma &= \mathcal{L}_\mathbb{I} \oplus \mathcal{L}_\varepsilon && \rightarrow \quad \text{non-invertible} \quad d_\sigma = \sqrt{2}
\end{aligned}$$

This is an example of a Tambara-Yamagami category

Example: Tambara-Yamagami categories

The structure of the Ising category can be generalized...

A Tambari-Yambari category arises from an abelian symmetry category $\mathcal{A}$

$$\mathcal{L}_a \otimes \mathcal{L}_b = \mathcal{L}_{a+b} \qquad \rightarrow \qquad d_a = 1$$

by extending it by a single non-invertible self-dual defect line $\mathcal{L}_M$

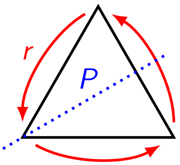
$$\mathcal{L}_a \otimes \mathcal{L}_M = \mathcal{L}_M$$

$$\mathcal{L}_M \otimes \mathcal{L}_M = \bigoplus_{a \in \mathcal{A}} \mathcal{L}_a \qquad \rightarrow \qquad d_M = \sqrt{|\mathcal{A}|}$$

One can think of this as a $\mathbb{Z}_2$-graded category.

Rep(S_3) symmetry is implemented by 3 simple defect lines, corresponding to the 3 irreducible representations of S_3.

Their fusion is the tensor product of these representations.
Their F-symbols are the 6j-symbols.



Representations:

- 1D trivial $\mathcal{L}_0 = \mathcal{L}_{\mathbb{I}}$
- 2D faithful $\mathcal{L}_1$
- 1D alternating $\mathcal{L}_2$

$\otimes$	$\mathcal{L}_1$	$\mathcal{L}_2$
$\mathcal{L}_1$	$\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$	$\mathcal{L}_1$
$\mathcal{L}_2$	$\mathcal{L}_1$	$\mathcal{L}_0$

The quantum dimensions coincide with the ordinary dimensions.

We note that this corresponds to the fusion in $SU(2)_4$ WZW models with $j = 0, 1, 2$ being the spin

Symmetry categories from CFT

Every "nice" (think "rational") CFT gives rise to a symmetry category via the modular fusion category associated with its primary fields

- Objects: Labels of primary fields
- Fusion: Fusion of primary fields
- F-symbols: Fusing matrices for conformal blocks

CFTs are a natural playground for the study of generalized symmetries

Remark: These categories have more structure than strictly required for a symmetry category (e.g. a braiding)

Generalized symmetries in 1+1D CFT

Some CFTs and their symmetry categories

Ising CFT

- Diagonal Virasoro minimal model with 3 primary fields
- Ising symmetry category
- $\mathrm{Vec}_{\mathbb{Z}_2}$ subsymmetry

Tricritical Ising CFT

- Diagonal Virasoro minimal model with 6 primary fields
- Fibonacci subsymmetry

3-state Potts model

- Diagonal W_3 minimal model with 6 primary fields
- Fibonacci subsymmetry

The $SU(2)_4$ WZW model (A-type invariant)

Field content

- Diagonal model with 5 primary fields
- These are labeled by spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \rightarrow$ truncation

Symmetry category

- Their fusion is given by

$$j_1 \otimes j_2 = \bigoplus_{j=|j_1-j_2|}^{\min(j_1+j_2, 4-j_1-j_2)} j$$

- There is a subcategory generated by integer spins $j = 0, 1, 2$
- This subcategory corresponds to $\mathrm{Rep}(S_3)$

Computation of the symmetry-resolved entanglement entropy in CFTs

Boundaries and defects in CFTs

We will consider diagonal rational 1+1D CFTs. Representations of its chiral symmetry algebra form a (modular) fusion category $\mathcal{M}$.

Goal: Symmetry-resolve entanglement wrt subcategories $\mathcal{C} \subset \mathcal{M}$.

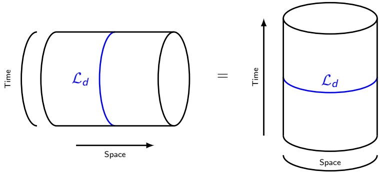
Verlinde lines $\mathcal{L}_d$ with $d \in \mathcal{M}$ [Petkova,Zuber'01]

- Preserve full chiral symmetry
- In particular: Are topological
- Satisfy $\mathcal{L}_c \otimes \mathcal{L}_d = \sum_e N_{cd}^e \mathcal{L}_e \quad \rightarrow \quad$ abelian fusion

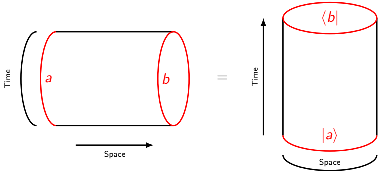
Cardy boundary states $|a\rangle$ with $a \in \mathcal{M}$ [Cardy'86]

- Preserve chiral symmetry
- Satisfy $\mathcal{L}_d|a\rangle = \sum_b N_{da}^b |b\rangle \quad \rightarrow \quad$ "regular representation"

Defect lines



Boundary states



In terms of partition functions this reads [Cardy'86] [Petkova,Zuber'01]

$$\mathrm{tr}_{\mathcal{H}_d}\left[q^{L_0-\frac{c}{24}}q^{\bar{L}_0-\frac{c}{24}}\right]=\mathrm{tr}_{\mathcal{H}}\left[\mathcal{L}_d\,\tilde{q}^{L_0-\frac{c}{24}}\tilde{q}^{\bar{L}_0-\frac{c}{24}}\right]\qquad\text{(defect)}$$

$$\mathrm{tr}_{\mathcal{H}_{ab}}\left[q^{L_0-\frac{c}{24}}\right]=\langle b|\,\tilde{q}^{L_0+\bar{L}_0-\frac{c}{12}}\,|a\rangle\qquad\text{(boundary states)}$$

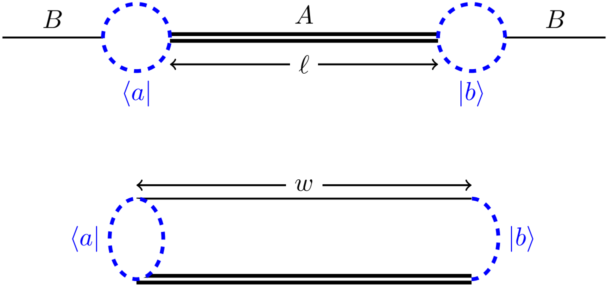
The LHS needs to have an expansion in terms of characters with integer coefficients and this leads to strong constraints on $\mathcal{L}_d$ and $|a\rangle$, respectively.

Why boundaries? Physics at the entangling surface

The computation of bipartite entanglement relies on the existence of a Hilbert space factorization

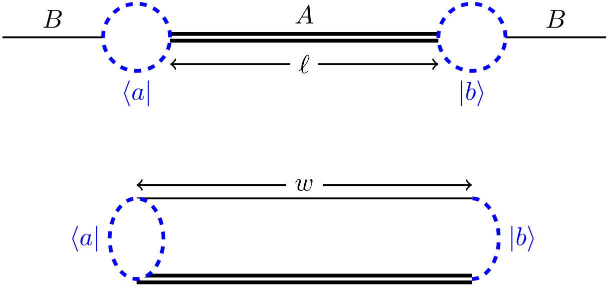
$$\mathcal{H} \longrightarrow \mathcal{H}_A \otimes \mathcal{H}_B$$

In a CFT, this requires a factorization map which depends on a choice of boundary conditions at the entanglement cut. [Ohmori,Tachikawa'15]



Mapping to BCFT

Bipartite geometries can be mapped to an annulus/CFT

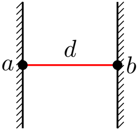


Entanglement spectrum	$\leftrightarrow$	**boundary CFT (in 1+1D)**
Symmetry resolution	$\leftrightarrow$	Rep theory on BCFT Hilbert space

[Cardy,Tonni'16]

Strongly and weakly symmetric boundary states

Symmetries are implemented by defect lines but not all defect lines can end on all boundary conditions.



$\Rightarrow$ a (resp b) needs to appear in the fusion of d with a (resp b)

One says that a boundary state $|b\rangle$ is weakly symmetric wrt $\mathcal{L}_d$ if

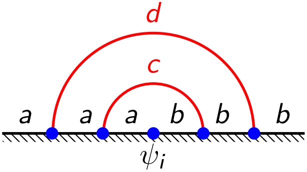
$$\mathcal{L}_d|b\rangle = |b\rangle \oplus \text{(possibly) other boundary states}$$

One says that a boundary state $|b\rangle$ is strongly symmetric wrt $\mathcal{L}_d$ if

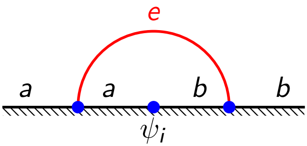
$$\mathcal{L}_d|b\rangle = \underbrace{\langle\mathcal{L}_d\rangle}_{\text{quantum dimension}} |b\rangle \qquad (|b\rangle \text{ is eigenstate})$$

[Choi,Rayhaun,Sanghavi,Shao'23]

The fusion rules are modified in the presence of boundaries.
They can even be non-integral [Kojita,Maccaferri,Masuda,Schnabl'16]



$$\sim \sum_{e\in c\otimes d} \tilde{N}_{cd}^{[a,b]e}$$



The coefficients $\tilde{N}_{cd}^{[a,b]e}$ determine the SREE.

Example: Fibonacci symmetry in tricritical Ising (see below)

Bulk fusion: $\mathcal{L}_\varepsilon \otimes \mathcal{L}_\varepsilon = \mathcal{L}_{\mathbb{I}} + \mathcal{L}_\varepsilon$

Boundary fusion: $\mathcal{L}_\varepsilon^{[aa]} \otimes \mathcal{L}_\varepsilon^{[aa]} = \frac{1}{\varphi}\,\mathcal{L}_{\mathbb{I}}^{[aa]} + \frac{\varphi-1}{\varphi}\,\mathcal{L}_\varepsilon^{[aa]}$ with $\varphi = \frac{1}{2}(1+\sqrt{5})$

Boundary CFT computation of the SREE (I)

The entanglement spectrum of a CFT can be identified with the spectrum of a boundary CFT. We thus have [Cardy,Tonni'16]

$$\rho_A = \frac{q^{L_0 - \frac{c}{24}}}{Z(q)} \qquad \text{where} \qquad Z(q) = \mathrm{tr}\left[q^{L_0 - \frac{c}{24}}\right]$$

where q encodes the geometry of the bipartition. With

$$\mathcal{Z}(q^n, r) = \mathrm{tr}\left[\Pi_r \rho_A^n\right]$$

one finds

$$S_n(q, r) = \frac{1}{1-n} \log \frac{\mathcal{Z}(q^n, r)}{\mathcal{Z}(q, r)^n} \qquad \text{and} \qquad S_{EE}(q, r) = \lim_{n \to 1} S_n(q, r)$$

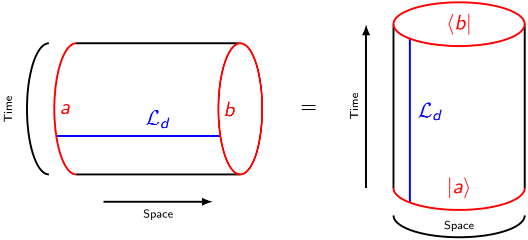
We are interested in evaluating these quantities for $q \to 1$

Boundary CFT computation of the SREE (II)

Symmetry-resolution also needs to take place in the boundary CFT

The computation follows this recipe:

- The limit $q \to 1$ is best evaluated in the closed string channel
- Express projectors Π_r as linear combinations of defect lines $\mathcal{L}_d$
- However, only the trivial defect will contribute



Warm-up: SREE for the finite group case

For finite groups, the projection operators are

$$\Pi_r = \frac{\chi_r(1)}{|G|} \sum_{g \in G} \chi_r(g)^* \, \mathcal{L}_g = \frac{d_r^2}{|G|} \, \mathcal{L}_{\mathbb{I}} + \text{other defects}$$

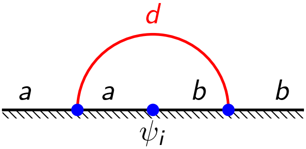
where $\mathcal{L}_g$ is the topological defect line implementing g

This leads to the deviation from entanglement equipartition

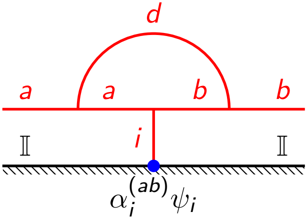
$$\Delta S_{EE}(q,r) = S_{EE}(q,r) - S_{EE}(q) \xrightarrow{q \to 1} \log \frac{d_r^2}{|G|}$$

[Kusuki,Murciano,Ooguri,Pal'23]

For defect d to topologically terminate on boundaries a and b, these must be weakly symmetric. The defect action is then [Kojita,Maccaferri,Masuda,Schnabl'16]

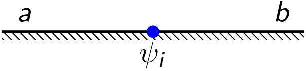


$\sim$



$\sim$ ...many F-moves...

$$= F_{ba}\left[\begin{pmatrix} a & d \\ i & b \end{pmatrix}\right]$$



The open string partition function is

$$\mathrm{tr}\left(q^{L_0-\frac{c}{24}}\right)=\sum_h N_{ab}^h\chi_h(q)$$

The projection onto the representation r,

$$\mathrm{tr}(P_r\rho_A)=N_{ab}^r\chi_r(q)\ ,$$

is implemented by the projector

$$P_r^{[a,b]}=\frac{1}{F_{1b}\begin{bmatrix}a & r\\ a & r\end{bmatrix}}\sum_s F_{1s}\begin{bmatrix}a & a\\ a & a\end{bmatrix}F_{sr}\begin{bmatrix}a & b\\ a & b\end{bmatrix}\mathcal{L}_s$$

In the limit $q\to 1$ we focus on the summand where $s=\mathbb{I}$

Final results

From the previous computations one can show that, as $q \to 1$,

$$\mathcal{Z}_{ab}(q,r) \sim \frac{d_r}{d_a d_b} \underbrace{\langle a|0\rangle\langle 0|b\rangle}_{\text{g-factors}} \tilde{q}^{\frac{c}{12}}$$

The prefactor is a product of quantum dimensions.

Finally one obtains the deviation from equipartition as

$$\Delta S_{EE}(q,r) = S_{EE}(q,r) - S_{EE}(q) \xrightarrow{q\to 1} \log \frac{d_r}{d_a d_b}$$

This result agrees with the (more general) expression obtained in [Choi,Rayhaun,Zheng'24] which relaxed some of our assumptions.

Their paper used the SymTFT approach and boundary tube algebras

[Choi,Rayhaun,Zheng'24] [Bhardwaj,Copetti,Pajer,Schäfer-Nameki'25

The tricritical Ising model has a modular category $\mathcal{M}$ with six simple objects, containing a Fibonacci subcategory $\mathcal{C} = \{\mathbb{I}, \varepsilon\}$.

Symmetry-resolution for this setup was first studied in [Saura-Bastida,Das,Sierra,Molina-Vilaplana '24]

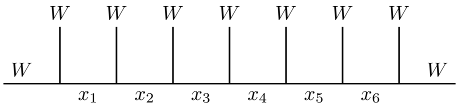
We choose the same weakly-symmetric boundary state and compute the breaking of equipartition to be

$$\Delta S_n(q,1) = \ln \frac{1}{1+\varphi}\ , \qquad \Delta S_n(q,\varepsilon) = \ln \frac{\varphi}{1+\varphi}$$

where φ is the golden ratio, finding a mismatch.

Numerical verification in anyonic chains

The possible states $(x_1, \ldots)$ of the golden chain are given by fusion trees



The interactions can be chosen to favor fusion $W \otimes W \to \mathbb{I}$. The resulting Hamiltonian is critical and corresponds to the tricritical Ising model with $c = \frac{7}{10}$.

[Feiguin,Trebst,Ludwig,Troyer,Kitaev,Wang,Freedman'07]

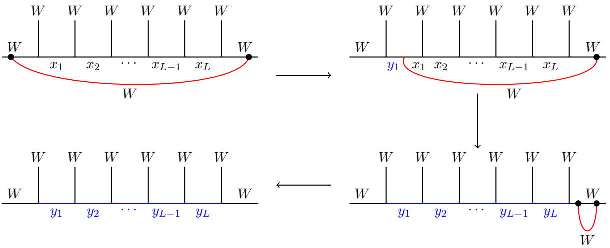
see also [Klümper, Pearce'92]

Technical remark

We use the golden chain as a proxy for the entanglement Hamiltonian

Exact diagonalization $\to$ entanglement spectrum

The action of topological defect lines can be implemented explicitly using the following topological symmetry defined in terms of F-moves

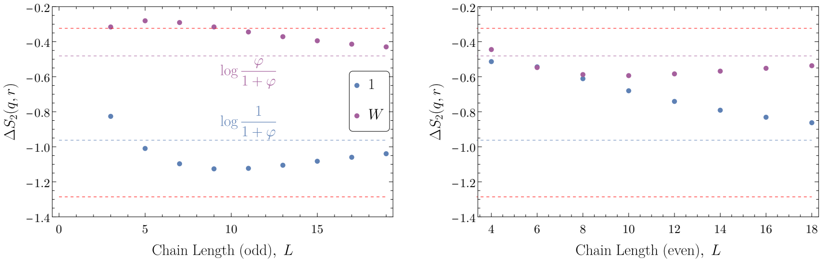


This results in

$$\mathcal{L}_W |x_1, \ldots, x_L\rangle = \sum_{y_1,\ldots,y_L} \prod_{n=0}^{L} F_{x_n y_{n+1}} \begin{bmatrix} y_n & W \\ W & x_{n+1} \end{bmatrix} |y_1, \ldots, y_L\rangle$$

This also permits to construct (and check) the projectors.

Numerical results



The red dashed lines show the result for a naive generalization of the finite group case to fusion categories:

$$\log \frac{\chi_r(1)^2}{|G|} \longrightarrow \log \frac{d_r^2}{1+\varphi^2} \qquad \text{(where } d_{\mathbb{I}} = 1 \text{ and } d_W = \varphi\text{)}$$

These simulations confirm our analytical results.

Summary and Outlook

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Summary

[Heymann,TQ'25]

We studied the symmetry resolution of the entanglement entropy for certain Verlinde lines in 1+1D see also [Das,Molina-Vilaplana,Saura-Bastida'24] [Choi,Rayhaun,Zheng'24]

The representation theory of boundary condition changing operators differs from that for the bulk for non-invertible symmetries see also [Choi,Rayhaun,Zheng'24]

Extensions and open problems concerning non-invertible symmetries

- $Rep(S_3)$ symmetry in $SU(2)_4$ [Heymann,TQ'25]
- Higher dimensions? for invertible symmetries: [Huang,Zhou'25]
- Entanglement asymmetry [Benini,Calabrese,Fossati,Singh,Venuti'25]
- Do boundary conditions matter? [Roy,Lukyanov,Saleur'25]
- Explicit realization in spin chains?
- RSOS models [Pearce,Heymann,TQ, in preparation]