On the integrability of the deformed Rule 54 model

Chiara Paletta

University of Ljubljana

Based on:

Ongoing work with T. Prosen

UNIVERZA V LJUBLJANI

11/11/2025

Challenges in Integrability, Wuhan

Overview

- Brief introduction on Rule 54 model: cellular automata – discrete time and space
- Introduction of Yang-Baxter integrability for quantum circuits and cellular automata
    - Deformations

Quantum: Strong evidence that the model is Yang-Baxter integrable

Stochastic: Boundary driven setup:

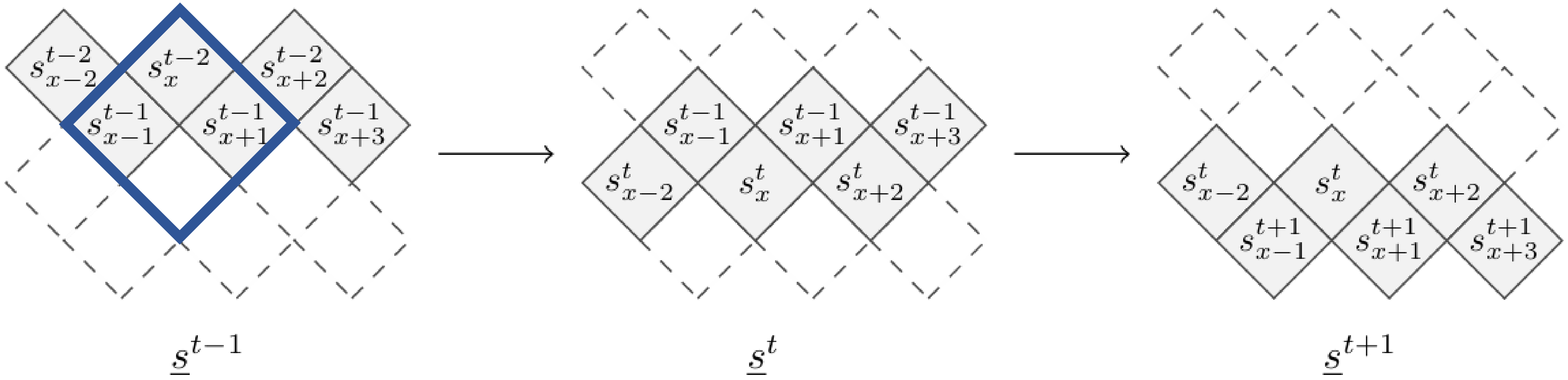
Exact solvability of the Non-Equilibrium Steady State

Take home message: Yang-Baxter integrability is more common than we think!

Model: Cellular automata

Simplest deterministic dynamics

[Review: B. Buča, K. Klobas, T. Prosen 2021]



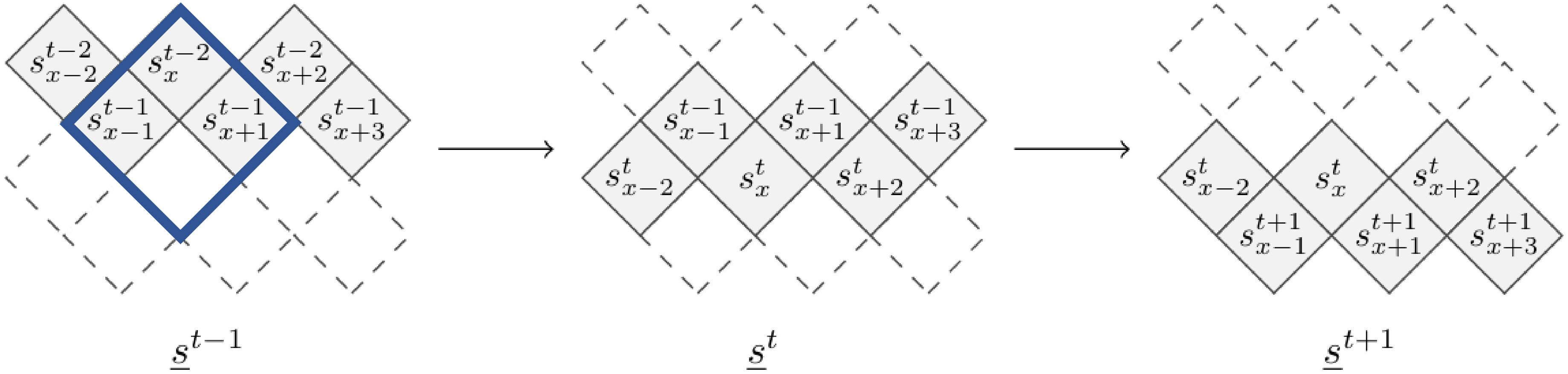
Dynamics: periodicity of 2 time steps

$$\mathbb{U} = \mathbb{U}_{\mathrm{o}}\mathbb{U}_{\mathrm{e}},$$
$$\mathbb{U}_{\mathrm{e}} = U_{123}U_{345}\dots U_{\mathrm{N-3,N-2,N-1}}U_{\mathrm{N-1,N,1}},$$
$$\mathbb{U}_{\mathrm{o}} = U_{234}U_{456}\dots U_{\mathrm{N-4,N-3,N-2}}U_{\mathrm{N-2,N-1,N}},$$

Model: Cellular automata

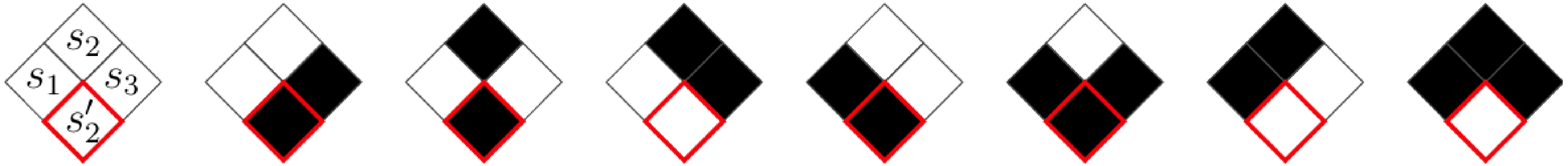
[Bobenko *et al.* (1993)]

[YB integrability proven by T. Gombor, B. Pozsgay]



Exactly solvable model: Super-integrable

Rule 54



$\lvert 000\rangle$	$\lvert 001\rangle$	$\lvert 010\rangle$	$\lvert 011\rangle$	$\lvert 100\rangle$	$\lvert 101\rangle$	$\lvert 110\rangle$	$\lvert 111\rangle$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\lvert 000\rangle$	$\lvert 011\rangle$	$\lvert 010\rangle$	$\lvert 001\rangle$	$\lvert 110\rangle$	$\lvert 111\rangle$	$\lvert 100\rangle$	$\lvert 101\rangle$

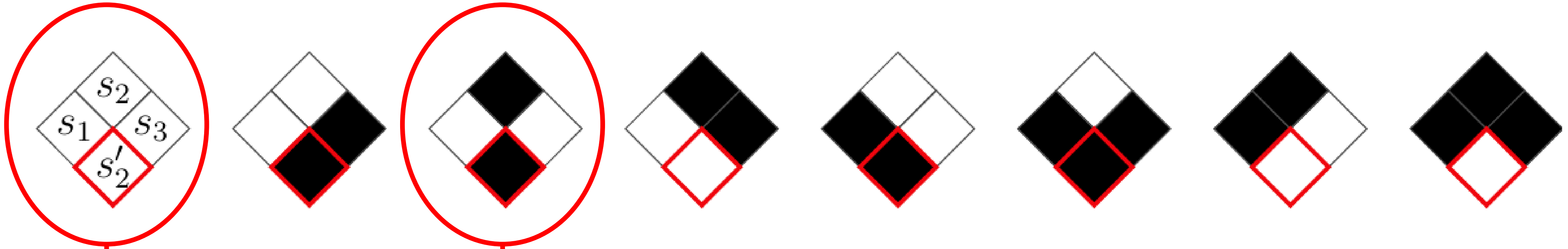
$$f_{00} = 1, \qquad f_{01} = f_{10} = f_{11} = X$$

NOT gate

Model: Cellular automata - Rule 54

+ Quantum deformation

[New model introduced by T. Prosen '21]



$$|000\rangle \to \alpha\,|000\rangle + \beta|010\rangle$$

$$|010\rangle \to \gamma\,|010\rangle + \delta|000\rangle$$

$$f_{00} = 1 \quad \Rightarrow \quad f_{00} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$f_{01} = f_{10} = f_{11} = X$$ NOT gate

By naively counting the charges and using numerical techniques, it seems that the deformed Rule 54 is integrable.

Can we prove it?

What does it mean for a cellular automata to be integrable?

$$\mathbb{U} = \mathbb{U}_o \mathbb{U}_e,$$
$$\mathbb{U}_e = U_{123} U_{345} \ldots U_{N-3,N-2,N-1} U_{N-1,N,1},$$
$$\mathbb{U}_o = U_{234} U_{456} \ldots U_{N-4,N-3,N-2} U_{N-2,N-1,N},$$

$$U = \begin{pmatrix} \alpha & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \gamma & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

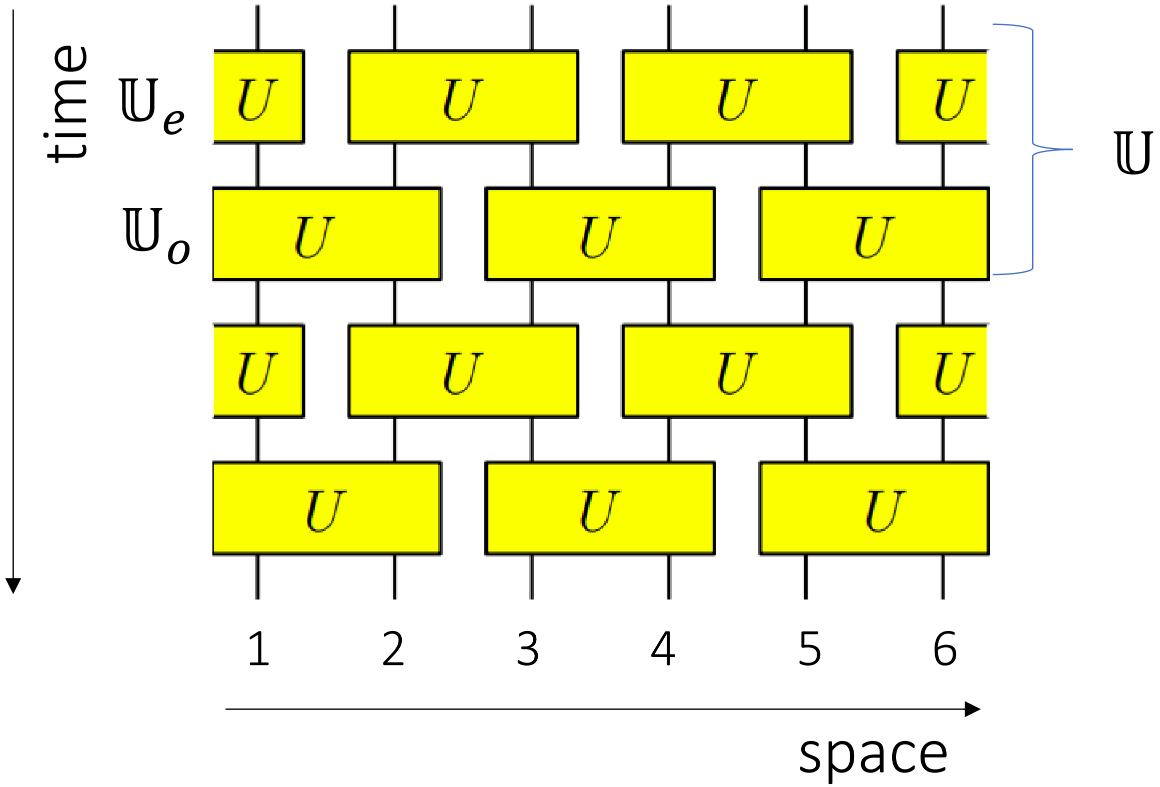
$\mathbb{U}$ commutes with an infinite amount of conserved charges

How can we prove it?

First, a short detour!

Integrability for quantum circuits

Let us consider for a moment a different setup: Brickwork type circuit



Write the dynamical evolution:

$$\mathbb{U} = \mathbb{U}_o \mathbb{U}_e = U_{12}\, U_{34} U_{56} U_{23}\, U_{45}\, U_{61}$$

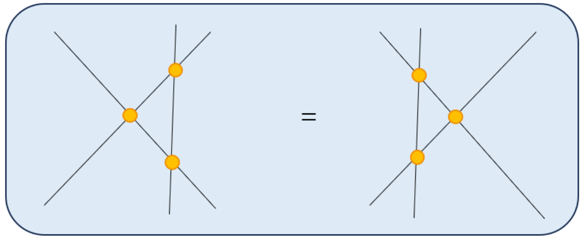
$$\mathbb{U}_e = U_{23}\, U_{45}\, U_{61}$$

$$\mathbb{U}_o = U_{12}\, U_{34}\, U_{56}$$

The circuit is integrable if the dynamical evolution generator $\mathbb{U}$ commutes with a tower of conserved charges.

Integrability for quantum circuits

A possible way to realize it is:



[Prosen, Zadnik, Vanicat PRL '18, Vanicat NPhys B '18]

The gate U is connected to the solution of the Yang-Baxter equation

$$U = \check{R}(2\kappa) = P\ R(2\kappa)$$

For a model characterized by an R-matrix solution of the YBE:

$$t(u) = \mathrm{Tr}_0 R_{06}(u-u_6)R_{05}(u-u_5)R_{04}(u-u_4)R_{03}(u-u_3)R_{02}(u-u_2)R_{01}(u-u_1)$$

$[t(u), t(v)] = 0$,   for all u and v

Can we fix the values of the inhomogeneities $u_1, \dots, u_L$ such that:

$$[\mathbb{U}, t(u)] = 0$$

YES! If

$$u_1 = u_3 = u_5 = -\kappa$$
$$u_2 = u_4 = u_6 = \kappa$$

Integrability for quantum circuits

[Prosen, Zadnik, Vanicat PRL '18, Vanicat NPhys B '18]

We have to find a way to write $\mathbb{U}$ as a function of $t(u)$:

$$t(u) = \mathrm{Tr}_0 R_{06}(u-\kappa)R_{05}(u+\kappa)R_{04}(u-\kappa)R_{03}(u+\kappa)R_{02}(u-\kappa)R_{01}(u+\kappa)$$

Let us compute $t(\kappa)$: Using $R(0) = P, \quad P\,R(2\,\kappa) = U, \quad P\,R = \check{R},$

$$t(\kappa) = \mathrm{Tr}_0 P_{06}P_{05}\check{R}_{05}(2\kappa)P_{04}P_{03}\check{R}_{03}(2\kappa)P_{02}P_{01}\check{R}_{01}(2\kappa) = \cdots$$
$$= P_{16}\;P_{15}P_{14}P_{13}P_{12}U_{23}U_{45}U_{16}$$

Using $\mathrm{Tr}_0 P_{0i} = 1$

Similarly $t(-\kappa)$:

$$R(-2\kappa) = R^{-1}(2\kappa) = U^{-1}$$

$$t(-\kappa) = P_{16}P_{15}P_{14}P_{13}P_{12}\;U_{56}^{-1}U_{34}^{-1}U_{12}^{-1}$$

$$\boxed{\mathbb{U} = t^{-1}(-\kappa)t(\kappa)}$$

since $[t(v), t(u)] = 0, \quad \forall\, u, v$

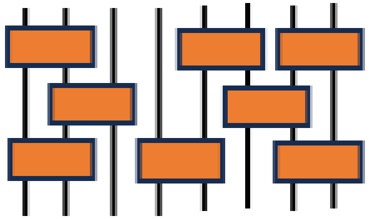
$\Rightarrow$

$$[\mathbb{U}, t(u)] = 0\,, \qquad \forall\, u$$

$t(u)$ generates the infinite tower of conserved charges commuting with $\mathbb{U}$

The circuit is Yang-Baxter integrable!

Given a quantum circuit with a geometry, can we understand if it is integrable?

Step 1: Draw the circuit	Step 2: Insert D	Step 3: Insert the expression of the quantum gate U	Step 4: Does the gate satisfy the YBE?	Step 5: Is the quantum circuits integrable?
	$D = 2$	$U = \frac{1}{\sqrt{1-u^2}} \begin{pmatrix} 1+u & 0 & 0 & 0 \\ 0 & 1 & u & 0 \\ 0 & u & 1 & 0 \\ 0 & 0 & 0 & 1+u \end{pmatrix}$	YES	YES! (or NO) $M = U_{12}U_{56}U_{78}U_{23}U_{67}$ $U_{12}U_{45}U_{78}$ The transfer matrix is: …

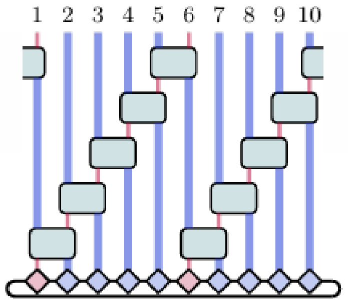
Dimension of local Hilbert space

Classification of nearest neighbour **integrable** circuits

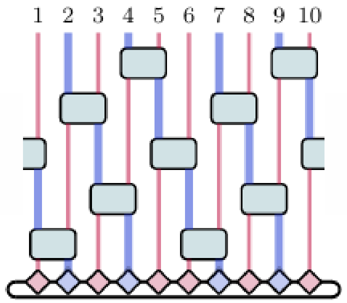
[Duh, CP, Pozsgay, Zadnik , 2503.04673]
[Miao, Gritsev, Kurlov, 2206.15142]

We proved that:
All quantum unitary circuits in which the gate (solution of the YBE) is applied to each pair of neighbouring qubits exactly once per period are integrable.

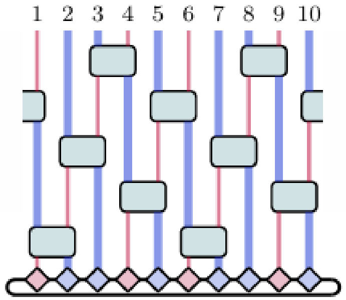
For example, for $L = 10$



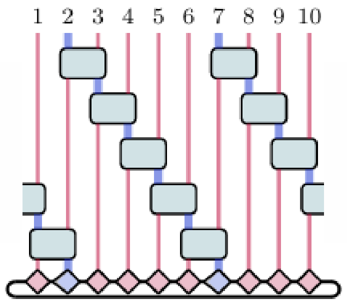
(a) $(q,r)=(5,1)$, $p=2$



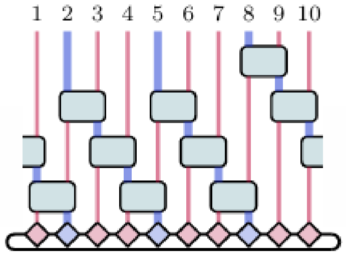
(b) $(q,r)=(5,2)$, $p=6$



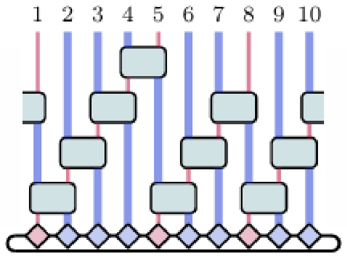
(c) $(q,r)=(5,3)$, $p=4$



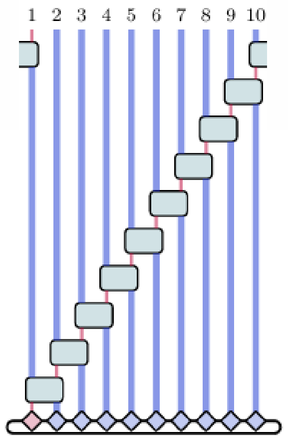
(d) $(q,r)=(5,4)$, $p=8$



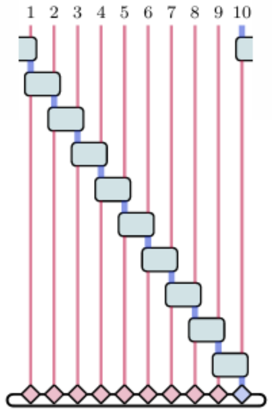
(e) $(q,r)=(10,3)$, $p=7$



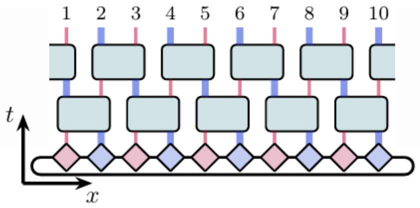
(f) $(q,r)=(10,7)$, $p=3$



(g) $(q,r)=(10,1)$, $p=1$



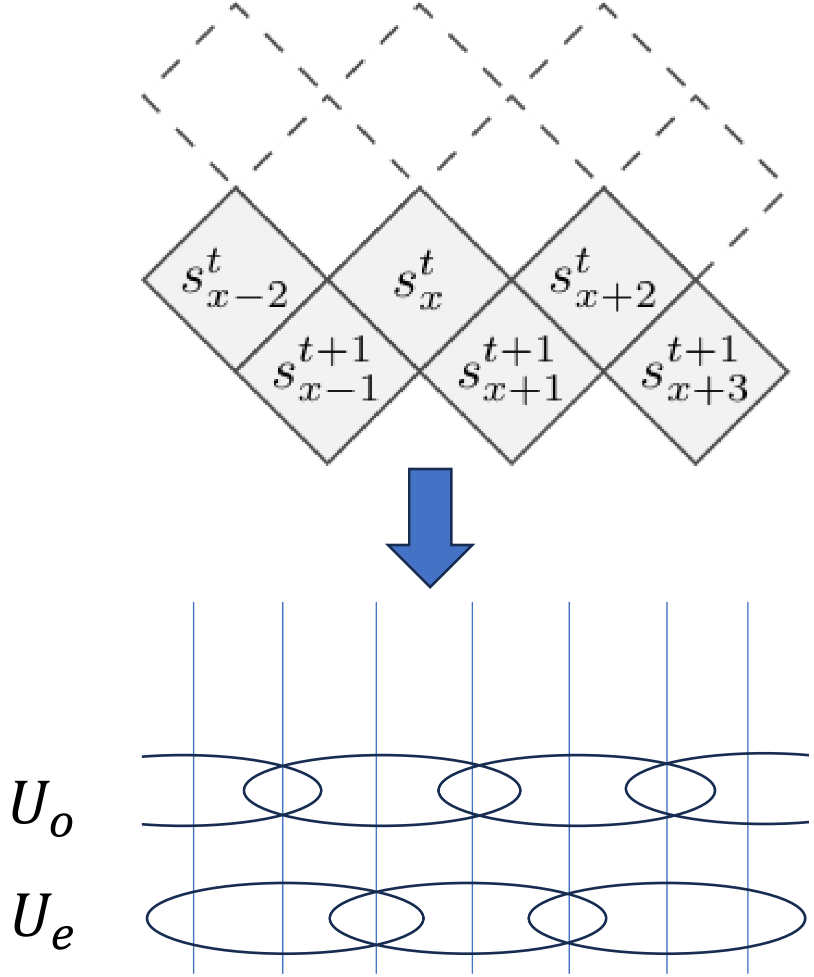
(h) $(q,r)=(10,9)$, $p=9$

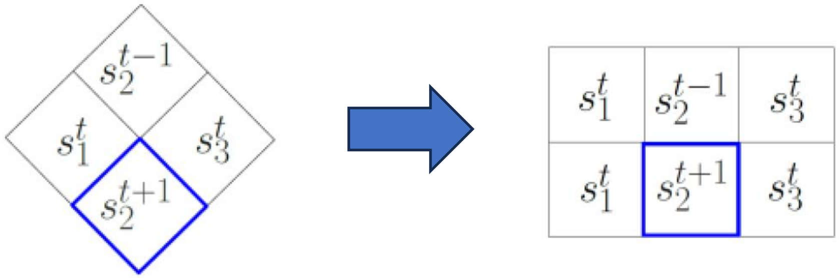


Let's go back to deformed Rule 54

[undeformed: T. Gombor, B. Pozsgay 2021]

Rule 54 (and the deformed version) as a quantum circuit





$$U = \begin{pmatrix} \alpha & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \gamma & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

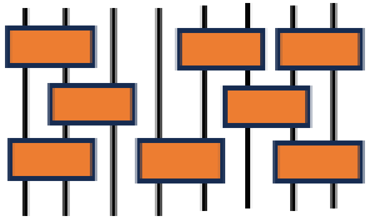
$$U_e = U_{L-1,L,1} \dots U_{3,4,5} U_{1,2,3}$$
$$U_o = U_{L,1,2} \dots U_{4,5,6} U_{2,3,4}.$$

Dynamical evolution described by: $U_{i,i+1,i+2}$

Let's go back to deformed Rule 54

$$U = \begin{pmatrix} \alpha & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \gamma & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Can we use the algorithm for U nearest-neighbor gate modified to next to NN?

Step 1: Draw the circuit	Step 2: Insert D	Step 3: Insert the expression of the quantum gate U	Step 4: Does the gate satisfy the YBE?	Step 5: Is the quantum circuits integrable?
	D = 2	$U = \frac{1}{\sqrt{1-u^2}} \begin{pmatrix} 1+u & 0 & 0 & 0 \\ 0 & 1 & u & 0 \\ 0 & u & 1 & 0 \\ 0 & 0 & 0 & 1+u \end{pmatrix}$	YES	YES! (or NO) M=$U_{12}U_{56}U_{78}U_{23}U_{67}$ $U_{12}U_{45}U_{78}$ The transfer matrix is: ...

Does the gate satisfy the YBE? → Q1) Does U solve the RLL relation?

Let’s go back to deformed Rule 54

[medium range integrability
T. Gombor, B. Pozsgay]

[de Leeuw, Retore]

Q1) Does U solve the RLL relation?

$$\mathbb{U} = \mathbb{U}_o \mathbb{U}_e,$$
$$\mathbb{U}_e = U_{123} U_{345} \ldots U_{N-3,N-2,N-1} U_{N-1,N,1},$$
$$\mathbb{U}_o = U_{234} U_{456} \ldots U_{N-4,N-3,N-2} U_{N-2,N-1,N},$$

$$U = \begin{pmatrix} \alpha & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \gamma & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$R_{A,B}(u,v)\mathcal{L}_{A,j}(u)\,\mathcal{L}_{B,j}(v) = \mathcal{L}_{B,j}(v)\mathcal{L}_{A,j}(u)R_{A,B}(u,v)$$

$$R_{ab,cd}(u,v)\mathcal{L}_{ab,j}(u)\,\mathcal{L}_{cd,j}(v) = \mathcal{L}_{cd,j}(v)\mathcal{L}_{ab,j}(u)R_{ab,cd}(u,v)$$

$$U_{abj} \overset{???}{=} \mathcal{L}_{ab,j}(u)\Big|_{u\to\xi_0}$$

A1) No!

Q2) Is there some hope that deformed Rule 54 is still Yang-Baxter integrable?

A2)YES!

Q2) Is there some hope that deformed Rule 54 is still Yang-Baxter integrable?

A2)YES!

We need to use a different strategy!

First, we check what is the range of the first conserved charges

$$[\mathbb{U}, Q_x] = 0,$$

$$Q_x = \sum_i q_{i,i+1,i+2,\dots,i+x-1}$$

we obtained $x = 6$

Open question: Why?

Does Q_6 belong to a tower of conserved charges of an integrable model?

We consider Q_6 as the first non-trivial charge of a spin chain.

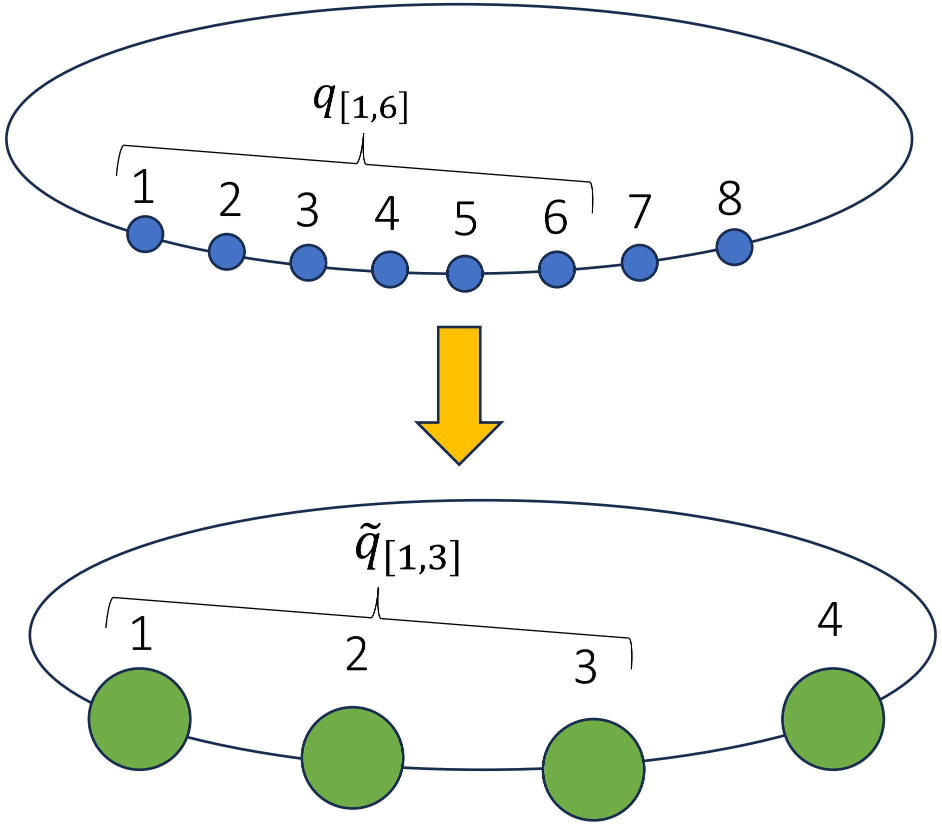
If we understand that:

$$\partial_u \ln t(u)\,|_{u\to 0} = Q_6 \;, \qquad [t(u), \mathbb{U}] = 0$$

$\Rightarrow$ Deformed Rule 54 is YB integrable!

Too complicated for this model!

We use the formalism of «medium» range spin chain



$$Q_6 = q_{[1,6]} + q_{[3,8]} + q_{[5,10]} + \cdots$$

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^4$$

$$Q_3 = \tilde{q}_{[1,3]} + \tilde{q}_{[2,4]} + \tilde{q}_{[3,5]} + \cdots$$

Is deformed Rule 54 YB integrable? $\longrightarrow$ Does Q_3 belong to a tower of conserved charges of an integrable model?

In the glued picture

[medium range integrability T. Gombor, B. Pozsgay]

[de Leeuw, Retore]

Suppose it exists a transfer matrix,

$$t(u) = \mathrm{Tr}_A \mathcal{L}_{a,b,L}(u)\, \mathcal{L}_{a,b,L-1}(u) \ldots \mathcal{L}_{a,b,1}(u)$$

the next conserved charge is related to the previous one as: $\tilde{q}_{i,i+1,i+2}$

$$Q_3 = \partial_u \log t(u) = \sum_i \tilde{q}_{i,i+1,i+2} \qquad Q_5 = \partial_u^2 \log t(u) = \sum_i [\tilde{q}_i, \tilde{q}_{i+1} + \tilde{q}_{i+2}] + w_{2i-1}$$

We checked that

$$[Q_3, Q_5] = 0$$

for an operator w_i

And

$$[Q_5, \mathbb{U}] = 0$$

This is a strong indication that deformed Rule 54 is YB integrable!

What do we need to have the full proof?

We know that:

$$[Q_3, \mathbb{U}] = 0$$

$$[Q_3, Q_5] = 0$$

$$[Q_5, \mathbb{U}] = 0$$

We need to show that:

1) $[Q_3, Q_i] = 0$, for all i

2) $[\mathbb{U}, Q_i] = 0$, for all i

$$\boxed{1)\ [Q_3, Q_i] = 0, \quad \text{for all } i}$$

Old conjecture: star-triangle hypothesis

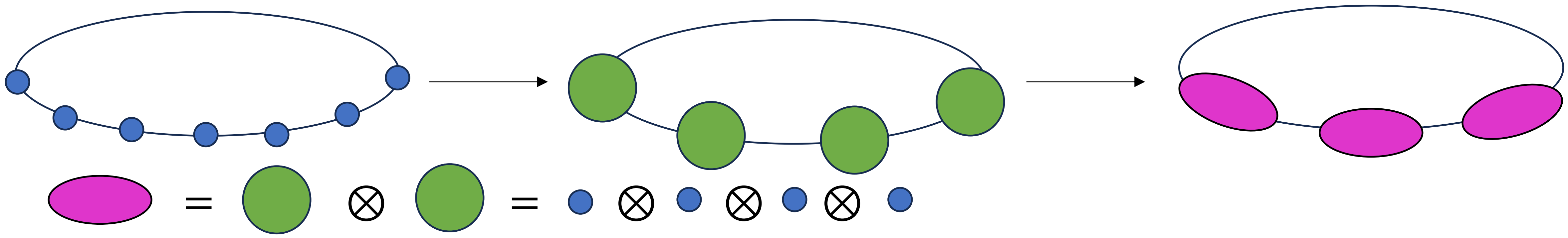
[M. Idzumi, T. Tokihiro, M. Arai]

Now proven by A. Hokkyo 2508.20713

Thanks to A. Hokkyo, we know that:

$$\begin{pmatrix} [Q_2, Q_3] = 0 \\ Q_3 = B\,[Q_2] \end{pmatrix} \Rightarrow \begin{pmatrix} [Q_2, Q_i] = 0, & \forall\, i \\ [Q_i, Q_j] = 0\,, & \forall\, i, j \end{pmatrix}$$

In our case, we have Q_3 and Q_5. We can glue the spaces again and consider



$$\tilde{\tilde{Q}}_2 = \sum_i \tilde{q}_{2i-1,2i,2i+1} + \tilde{q}_{2i,2i+1,2i+2}$$

Range 2 charge in the bigger Hilbert space

Local density: $\left(\tilde{\tilde{q}}_2\right)_{(1,2),(3,4)} = \tilde{q}_{123} + \tilde{q}_{234}$

We can now use the theorem by Hokkyo.

2) $[\mathbb{U}, Q_i] = 0$, for all i

The proof is different than 1) (done by Hokkyo)
Here, we cannot use the fact that the charges are translational invariant and local.

This is work in progress → Technical proof: ask me for more details.

We have: strong evidence of integrability

We have also obtained the expression of the next conserved charge:

$$\tilde{q}^{(7)} = \left[\tilde{q}_{i+4} + \tilde{q}_{i+3} + \frac{1}{2}\tilde{q}_{i+2}, [\tilde{q}_i + \tilde{q}_{i+1}, \tilde{q}_{i+2}]\right] - [\tilde{q}_{i+4}, w_{i+2} + w_{i+3}] + \frac{1}{2}[\tilde{q}_{i+2} + \tilde{q}_{i+3}, w_{i+4}] + y_i$$

We can solve: $[Q_{14}, Q_6] = 0$ For an operator y_i

Extra check

$$[Q_{14}, Q_{10}] = 0 \qquad [Q_{14}, \mathbb{U}] = [Q_{10}, \mathbb{U}] = 0$$

Strong evidence of integrability

$$\check{\mathcal{L}}(u) = 1 + u\, q_i + \frac{u^2}{2}\left(w_i + q_i^2\right) + \frac{u^3}{3!}\left(q_i^3 + 3q_i\left(-q_i^2 + w_i\right) + y_i\right) + O\,(u^3)$$

Strong indication that the model is Yang-Baxter integrable!

Complete proof would involve:

A) getting the higher conserved charges and resum!

In our case it is hard because the Lax operator has dimension $4^3 \times 4^3$

[de Leeuw, Retore "Lifting integrable models"]

[Gombor "Wrapping corrections"]

B) Completing the proof that $[\mathbb{U}, Q_i] = 0$, for all i

[Paletta, Prosen In progress]

Take home message: Yang-Baxter integrability is more common than we think!

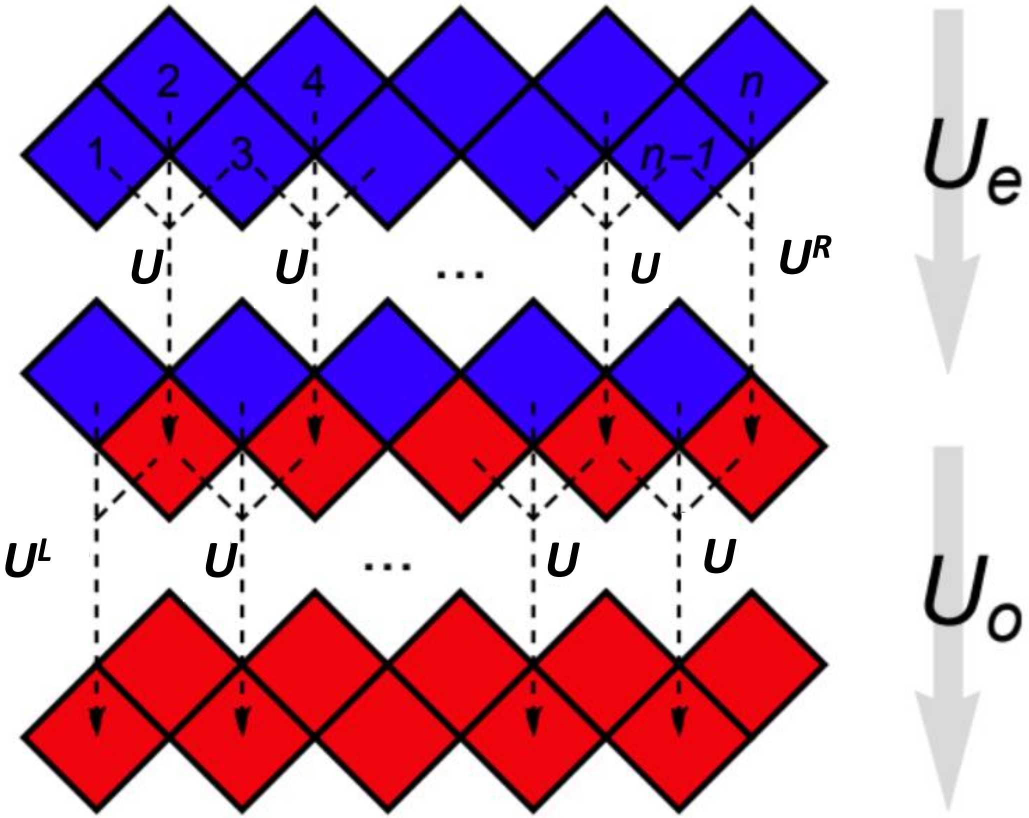
Boundary driven set-up

Integrability = exact solvability of the steady state!

$$f_{00} = \begin{pmatrix} \alpha & 1-\beta \\ 1-\alpha & \beta \end{pmatrix}$$

Stochastic deformation

$$f_{01} = f_{10} = f_{11} = X$$



Boundary: Conditional driving

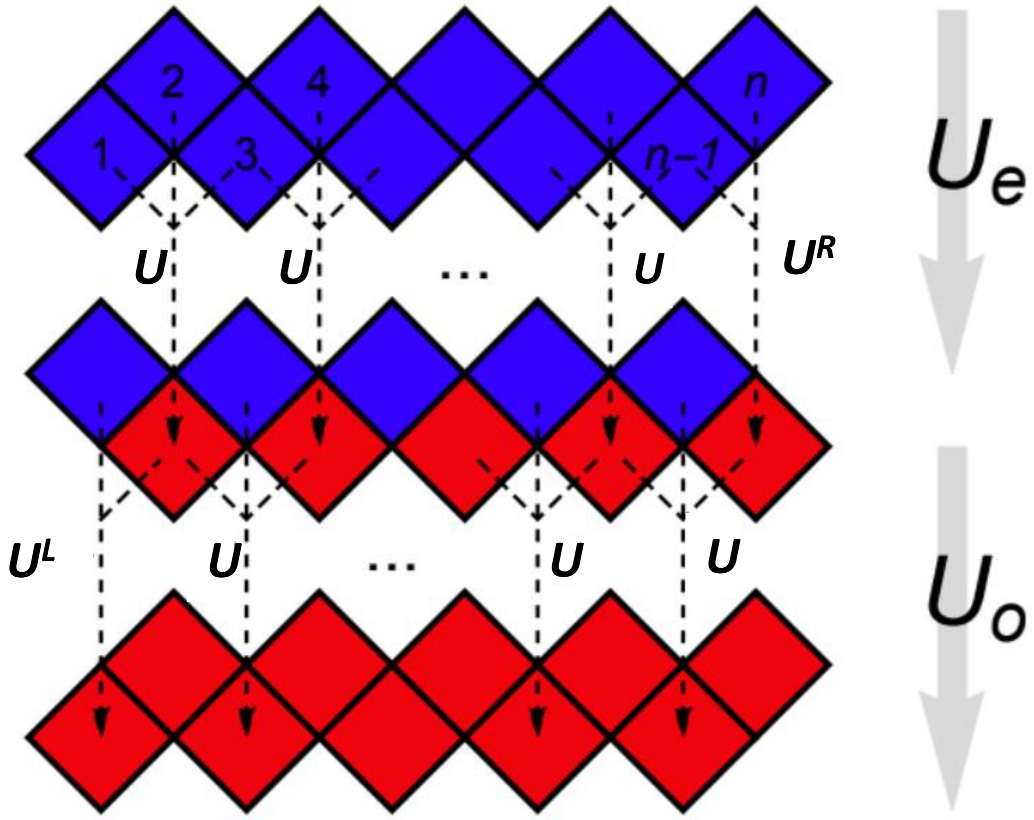
$$\mathbb{U} = \mathbb{U}_o \mathbb{U}_e,$$

$$\mathbb{U}_e = U_{123} U_{345} \dots U_{N-3,N-2,N-1} U^R_{N-1,N},$$

$$\mathbb{U}_o = U^L_{12} U_{234} U_{456} \dots U_{N-4,N-3,N-2} U_{N-2,N-1,N},$$

Boundary driven set-up

$$f_{00} = \begin{pmatrix} \alpha & 1-\beta \\ 1-\alpha & \beta \end{pmatrix}, \qquad f_{01} = f_{10} = f_{11} = X$$



$$U^L = \begin{pmatrix} a & 0 & a & 0 \\ 0 & b & 0 & b \\ 1-a & 0 & 1-a & 0 \\ 0 & 1-b & 0 & 1-b \end{pmatrix}$$

Conditional driving

$$U^R = \begin{pmatrix} c & c & 0 & 0 \\ 1-c & 1-c & 0 & 0 \\ 0 & 0 & d & d \\ 0 & 0 & 1-d & 1-d \end{pmatrix}$$

For example:

Locally

$$U^L|1\,1\rangle = a\,|1\,1\rangle + (1-a)|0\,1\rangle$$

$$U^L|1\,0\rangle = b\,|1\,0\rangle + (1-\mathrm{b})|0\,0\rangle$$

...

Theorem: For this form of $\mathbb{U}$ the steady state exists and it is unique.

$$0 < a, b, c, d < 1, 0 \le \alpha, \beta \le 1$$

Undeformed: Prosen, Mejia-Monasterio '16

Steady State solution:

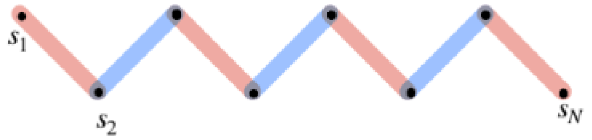
Matrix Product/Patch State Ansatz

$$\mathbb{U}\,\mathbf{f} = \mathbb{U}_\mathrm{o}\mathbb{U}_\mathrm{e}\,\mathbf{f} = \mathbf{f},$$

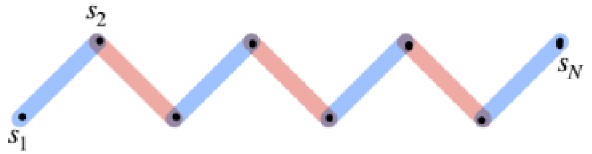
$$\mathbb{U}_\mathrm{e}\,\mathbf{f} = \mathbf{f}',$$

$$\mathbb{U}_\mathrm{o}\,\mathbf{f}' = \mathbf{f},$$

$$\mathbf{f} = \mathbf{L}_{12}\mathbf{Z}'_{23}\mathbf{Z}_{34}\mathbf{Z}'_{45}\mathbf{Z}_{56}\ldots\mathbf{Z}_{\mathrm{N-3,N-2}}\mathbf{Z}'_{\mathrm{N-2,N-1}}\mathbf{R}_{\mathrm{N-1,N}},$$

$=$ 

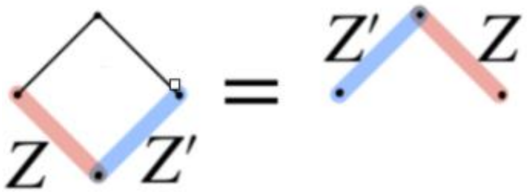
$$\mathbf{f}' = \mathbf{L}'_{12}\mathbf{Z}_{23}\mathbf{Z}'_{34}\mathbf{Z}_{45}\mathbf{Z}'_{56}\ldots\mathbf{Z}'_{\mathrm{N-3,N-2}}\mathbf{Z}_{\mathrm{N-2,N-1}}\mathbf{R}'_{\mathrm{N-1,N}}$$

$=$ 

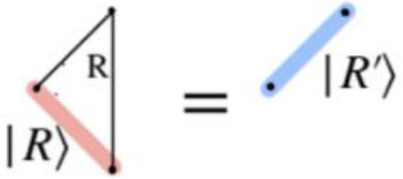
Steady State solution:

Bulk EoM: patch-state analogues of Zamolodchikov-Faddeev relations
Boundary EoM: Ghoshal-Zamolodchikov relations

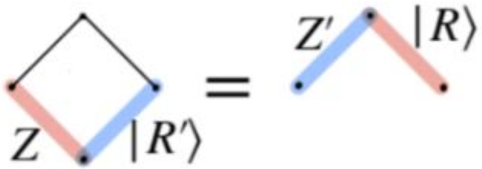
[Vanicat NPB'18]



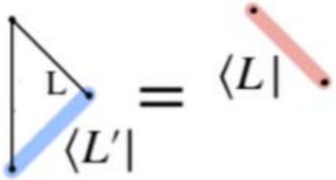
$$U_{123}\mathbf{Z}_{12}\mathbf{Z}'_{23} = \mathbf{Z}'_{12}\mathbf{Z}_{23},$$



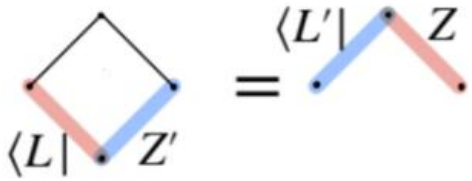
$$U^R_{12}\,\mathbf{R}_{12} = \mathbf{R}'_{12}$$



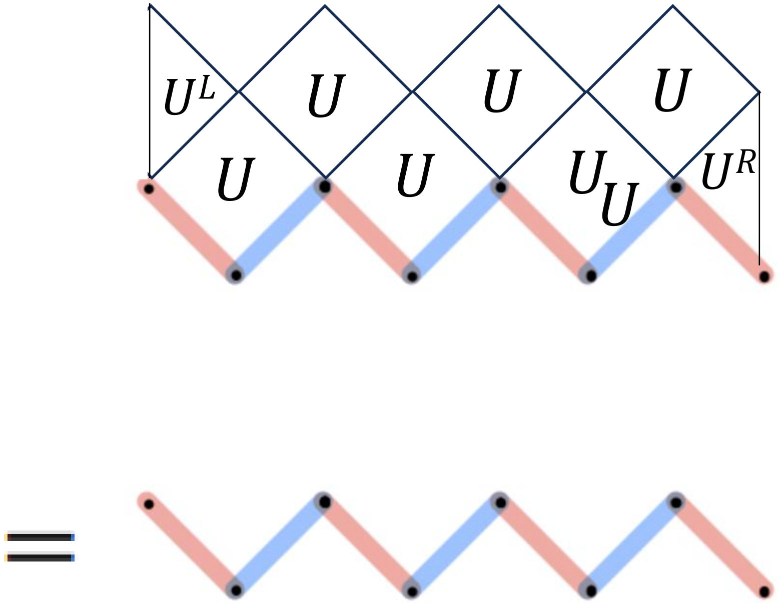
$$U_{123}\mathbf{Z}_{12}\mathbf{R}'_{23} = \mathbf{Z}'_{12}\mathbf{R}_{23}$$



$$U^L_{12}\mathbf{L}'_{12} = \mathbf{L}_{12}$$



$$U_{123}\mathbf{L}_{12}\mathbf{Z}'_{23} = \mathbf{L}'_{12}\mathbf{Z}_{23}$$



Steady State solution:

Construct explicit representation of Z, Z’, L, R, L’, R’

Bulk:

$$
\begin{aligned}
&(1-\gamma) Z_{00} Z'_{00} + \beta\, Z_{01} Z'_{10} = Z'_{00} Z_{00},\\
&\gamma\, Z_{00} Z'_{00} + (1-\beta) Z_{01} Z'_{10} = Z'_{01} Z_{10},\\
&Z_{s,s'} Z'_{s',s''} = Z_{s,\bar{s}'} Z'_{\bar{s}',s''}, \quad (s,s'') \neq (0,0),
\end{aligned}
$$

$$\bar{s} \equiv 1 - s.$$

We obtain that Z, Z′ are infinite dimensional and block-tridiagonal, the blocks are 3x3

$$
Z_{s,s'} = \begin{pmatrix}
Z^{(0,0)}_{s,s'} & Z^{(1,+)}_{s,s'} & & & \\
Z^{(1,-)}_{s,s'} & Z^{(1,0)}_{s,s'} & Z^{(2,+)}_{s,s'} & & \\
 & Z^{(2,-)}_{s,s'} & Z^{(2,0)}_{s,s'} & & \\
 & & & \dots & \\
\end{pmatrix}
$$

Steady State solution:

Construct explicit representation of Z, Z’, L, R, L’, R’

$$Z_{s,s'} = \begin{pmatrix} Z^{(0,0)}_{s,s'} & Z^{(1,+)}_{s,s'} & & & \\ Z^{(1,-)}_{s,s'} & Z^{(1,0)}_{s,s'} & Z^{(2,+)}_{s,s'} & & \\ & Z^{(2,-)}_{s,s'} & Z^{(2,0)}_{s,s'} & \; & \\ & & \; & \ldots & \end{pmatrix}$$

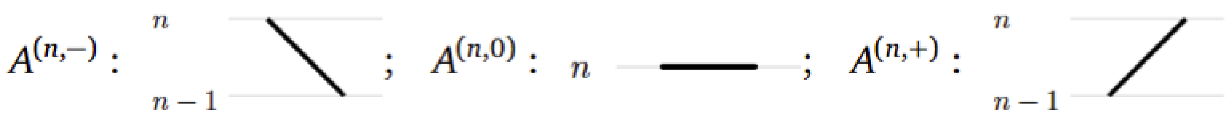
$$\sum_{t_1,t_2,t_3=0}^{1} (U)_{4s_1+2s_2+s_3+1,4t_1+2t_2+t_3+1} Z^{(n,-)}_{t_1,t_2} Z'^{(n-1,-)}_{t_2,t_3} = Z'^{(n,-)}_{s_1,s_2} Z^{(n-1,-)}_{s_2,s_3},$$

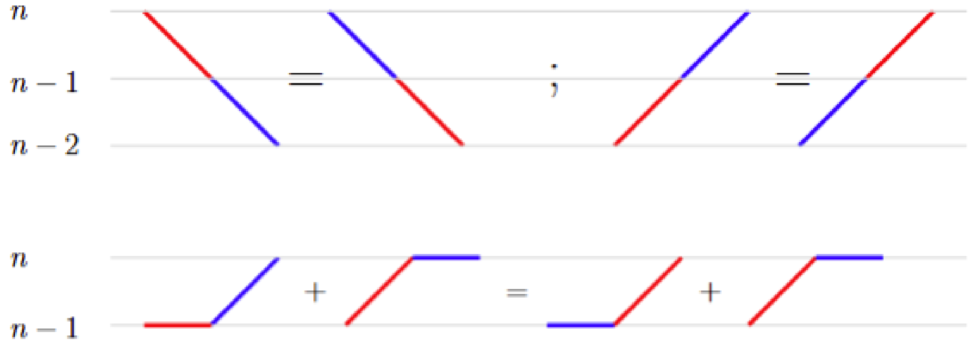
$$\sum_{t_1,t_2,t_3=0}^{1} (U)_{4s_1+2s_2+s_3+1,4t_1+2t_2+t_3+1} Z^{(n-1,+)}_{t_1,t_2} Z'^{(n,+)}_{t_2,t_3} = Z'^{(n-1,+)}_{s_1,s_2} Z^{(n,+)}_{s_2,s_3},$$

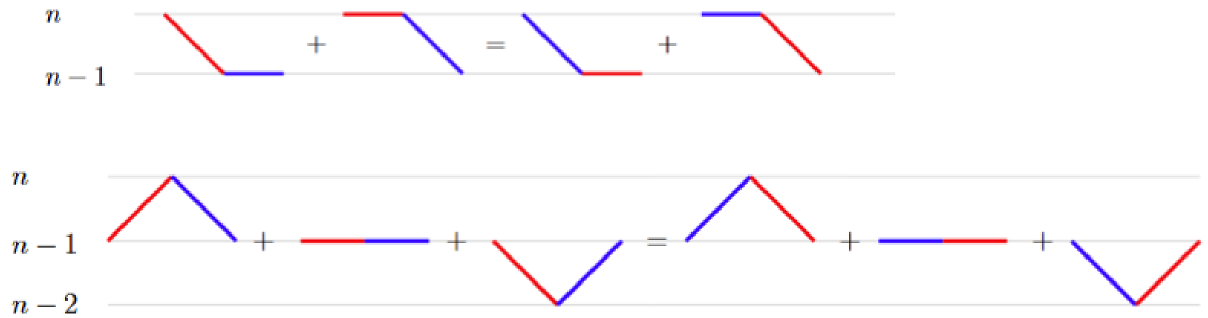
$$\sum_{t_1,t_2,t_3=0}^{1} (U)_{4s_1+2s_2+s_3+1,4t_1+2t_2+t_3+1} \left[Z^{(n-1,0)}_{t_1,t_2} Z'^{(n,+)}_{t_2,t_3} + Z^{(n,+)}_{t_1,t_2} Z'^{(n,0)}_{t_2,t_3}\right] = Z'^{(n-1,0)}_{s_1,s_2} Z^{(n,+)}_{s_2,s_3} + Z'^{(n,+)}_{s_1,s_2} Z^{(n,0)}_{s_2,s_3}$$

$$\sum_{t_1,t_2,t_3=0}^{1} (U)_{4s_1+2s_2+s_3+1,4t_1+2t_2+t_3+1} \left[Z^{(n,-)}_{t_1,t_2} Z'^{(n-1,0)}_{t_2,t_3} + Z^{(n,0)}_{t_1,t_2} Z'^{(n,-)}_{t_2,t_3}\right] = Z'^{(n,-)}_{s_1,s_2} Z^{(n-1,0)}_{s_2,s_3} + Z'^{(n,0)}_{s_1,s_2} Z^{(n,-)}_{s_2,s_3}$$

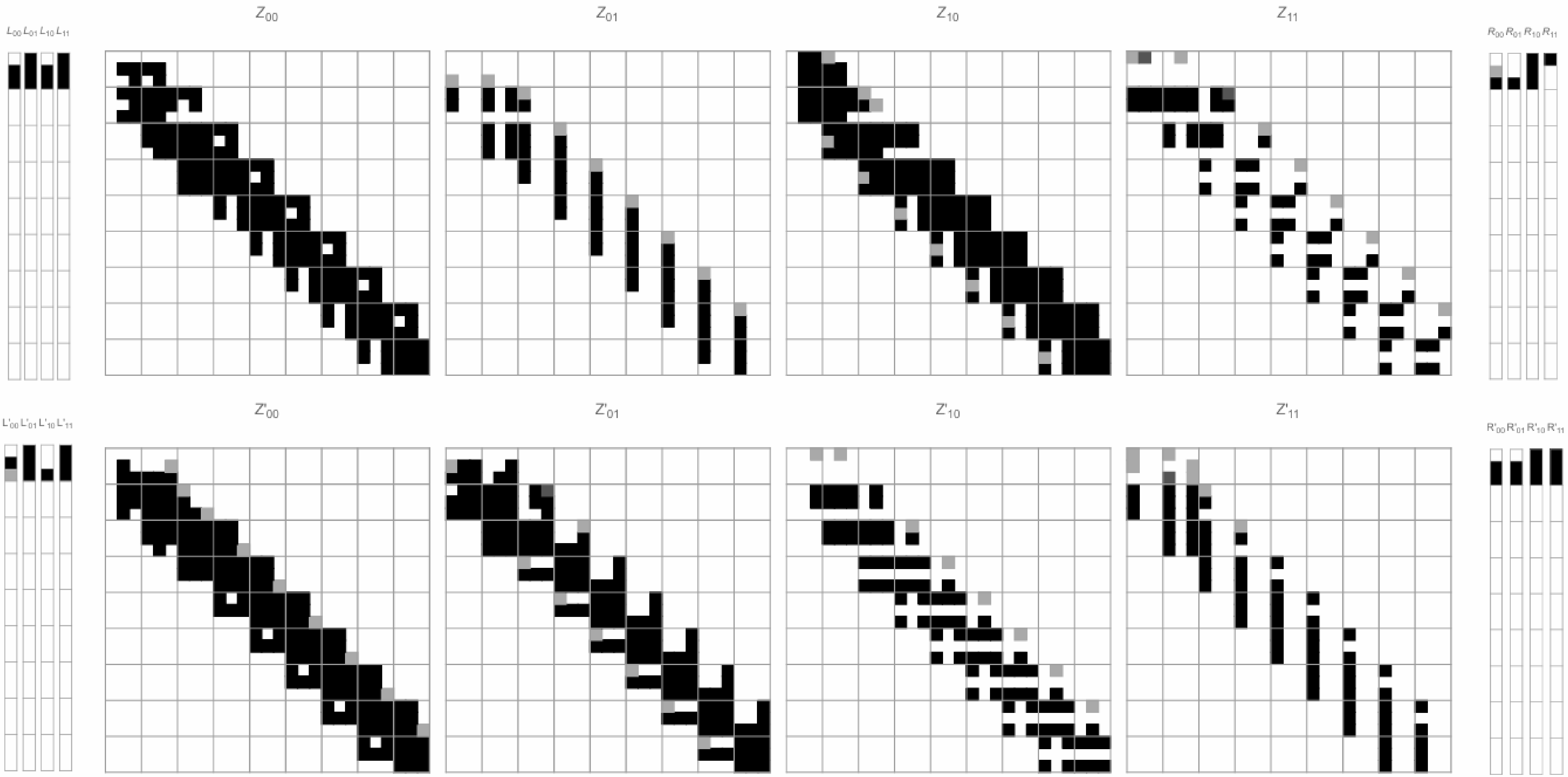
$$\sum_{t_1,t_2,t_3=0}^{1} (U)_{4s_1+2s_2+s_3+1,4t_1+2t_2+t_3+1} \left[Z^{(n,+)}_{t_1,t_2} Z'^{(n,-)}_{t_2,t_3} + Z^{(n-1,0)}_{t_1,t_2} Z'^{(n-1,0)}_{t_2,t_3} + Z^{(n-1,-)}_{t_1,t_2} Z'^{(n-1,+)}_{t_2,t_3}\right] =$$
$$Z'^{(n,+)}_{s_1,s_2} Z^{(n,-)}_{s_2,s_3} + Z'^{(n-1,0)}_{s_1,s_2} Z^{(n-1,0)}_{s_2,s_3} + Z'^{(n-1,-)}_{s_1,s_2} Z^{(n-1,+)}_{s_2,s_3}$$







Matrix product structure of the non-equilibrium steady state



Conclusions

- Strong perturbative evidence that a deformation of Rule 54 model is Yang-Baxter integrable
- Stochastic deformation + open boundary condition -> NESS

Open questions

- Just from the expression of $\mathbb{U}$, can we understand that the lowest conserved charge has interaction range 6?
- The transfer matrix commutes with $\mathbb{U}$, but how to generate $\mathbb{U}$ from it?
- How do we prove that for the boundary driven setup the full spectrum is also integrable? → Related work with A. Retore and M. de Leeuw

  Similar phenomena observed by: V. Popkov, T. Prosen 2502.06731
- Use tQ functional equations to determine algebraic integrability → With D. Fioravanti

Conclusions

- Strong perturbative evidence that a deformation of Rule 54 model is Yang-Baxter integrable
- Stochastic deformation + open boundary condition -> NESS

Open questions

- Just from the expression of $\mathbb{U}$, can we understand that the lowest conserved charge has interaction range 6?
- The transfer matrix commutes with $\mathbb{U}$, but how to generate $\mathbb{U}$ from it?
- How do we prove that for the boundary driven setup the full spectrum is also integrable? → Related work with A. Retore and M. de Leeuw

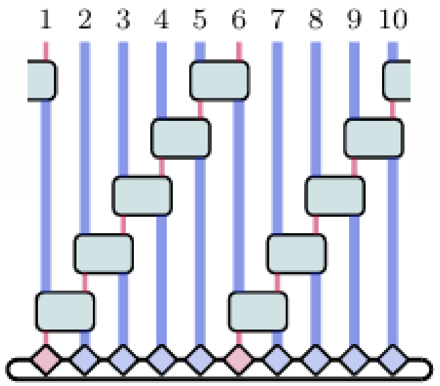
  Similar phenomena observed by: V. Popkov, T. Prosen 2502.06731
- Use tQ functional equations to determine algebraic integrability → With D. Fioravanti

Thanks!

Classification of nearest neighbour **integrable** circuits

We proved that:
All quantum unitary circuits in which the gate (solution of the YBE) is applied to each pair of neighbouring qubits exactly once per period are integrable.

For example, for $L = 10$



(a) $(q,r)=(5,1)$, $p=2$

$$\overrightarrow{u} = (\tau_1, \tau_2, \tau_2, \tau_2, \tau_2, \tau_1, \tau_2, \tau_2, \tau_2, \tau_2)$$

$$M = U_{56}\, U_{10,1} U_{45}\, U_{9,10} U_{34}\, U_{89} U_{23} U_{78}\, U_{12} U_{67} \quad ,$$

$$t(u) = Tr_0 \overleftarrow{\prod_{i=1}^{10}} R_{0i}(u - u_i)$$

$$M = t^{-1}(\tau_1) t(\tau_2)$$