Three-point functions in critical loop models

Jesper Lykke Jacobsen [1,2,3]

[1] 'Ecole Normale Supérieure, Paris

[2] Sorbonne Université, Paris

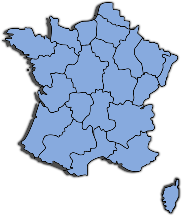
[3] Commisariat à l'Energie Atomique, Saclay

International workshop on “Challenges in Integrability”,
Wuhan, 18/11/2025

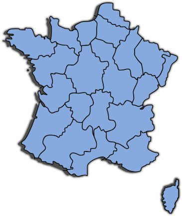
Collaborators:
M. Downing, L. Gräns-Samuelsson, Y. Ikhlef,
R. Nivesvivat, S. Ribault, P. Roux, H. Saleur


Heilongjiang
Jilin
Liaoning
Xinjiang
Inner Mongolia (Nei Mongol)
Beijing
Hebei
Tianjin
Ningxia
Shanxi
Shandong
Qinghai
Gansu
Tibet (Xizang)
Shaanxi
Henan
Jiangsu
Shanghai
Anhui
Hubei
Sichuan
Chongqing
Zhejiang
Guizhou
Hunan
Jiangxi
Fujian
Yunnan
Guangxi
Guangdong
Taiwan
(claimed)
Hong Kong
Macau
Hainan

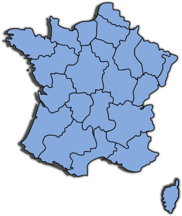

Heilongjiang
Jilin
Liaoning
Xinjiang
Inner Mongolia (Nei Mongol)
Beijing
Hebei
Tianjin
Ningxia
Shanxi
Qinghai
Shandong
Gansu
Tibet (Xizang)
Shaanxi
Henan
Jiangsu
Shanghai
Hubei
Anhui
Sichuan
Chongqing
Zhejiang
Guizhou
Hunan
Jiangxi
Fujian
Yunnan
Guangxi
Guangdong
Taiwan
(claimed)
Hong Kong
Macau
Hainan




Heilongjiang
Jilin
Liaoning
Xinjiang
Inner Mongolia (Nei Mongol)
Beijing
Hebei
Tianjin
Ningxia
Shanxi
Qinghai
Shandong
Gansu
Tibet (Xizang)
Shaanxi
Henan
Jiangsu
Hubei
Anhui
Shanghai
Sichuan
Chongqing
Zhejiang
Guizhou
Hunan
Jiangxi
Fujian
Yunnan
Guangxi
Guangdong
Taiwan
(claimed)
Hong Kong
Macau
Hainan

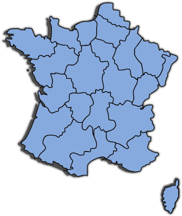



Heilongjiang
Jilin
Liaoning
Xinjiang
Inner Mongolia (Nei Mongol)
Beijing
Hebei
Tianjin
Ningxia
Shanxi
Shandong
Qinghai
Gansu
Shaanxi
Henan
Jiangsu
Tibet (Xizang)
Shanghai
Hubei
Anhui
Sichuan
Chongqing
Zhejiang
Guizhou
Hunan
Jiangxi
Fujian
Yunnan
Guangxi
Guangdong
Taiwan
(claimed)
Hong Kong
Macau
Hainan



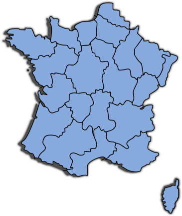
$9\,595\,961$ km^2

9 595 961 km^2



643 801 km^2

$9\,595\,961$ km^2



$643\,801$ km^2

$42\,925$ km^2

Tale of two loop models

Q-state Potts model

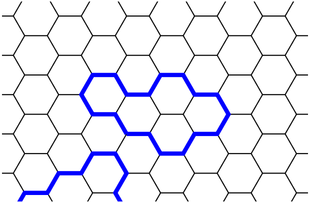
- Q-state spins; interactions have S_Q permutation symmetry.
- Equivalent loop model on medial lattice [Baxter-Kelland-Wu '76].
- Respects fixed orientation of lattice edges: $PSU(n)$ symmetry.
- Related to integrable 6-vertex model and Temperley-Lieb algebra.
- S_Q commutes with partition algebra $\mathscr{P}_L(Q)$, descending to Potts–Temperley–Lieb algebra $P\mathscr{TL}_{2L}(\sqrt{Q})$ in $d = 2$.

$O(n)$ model

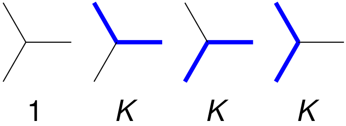
- Vector spins $\in \mathbb{R}^n$; interactions have $O(n)$ symmetry.
- Equivalent loop model in $d = 2$ after modification [Nienhuis '82].
- Related to integrable 19-vertex model and Motzkin algebra.
- $O(n)$ commutes with Brauer algebra $\mathscr{B}_L(n)$, descending to unoriented Jones–Temperley–Lieb algebra $u\mathscr{JTL}_L(n)$ in $d = 2$.

$O(n)$ model [Nienhuis '82]

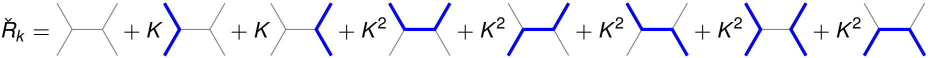
loop weight n



Vertex weights 1 and K



All configurations can be built by a transfer matrix:



Define the partition function

$$Z(K, n) = \sum_{\text{loops}} K^{\#\text{monomers}} n^{\#\text{loops}} .$$

$$Z(K,n) = \sum_{\text{loops}} K^{\#\text{monomers}} n^{\#\text{loops}} .$$

Monomer fugacity at the critical point:

$$K_c = \left(2 \pm \sqrt{2-n}\right)^{-1/2} ,$$

where $-2 \leq n \leq 2$. Plus (minus) sign for the dilute (dense) phase.

Special cases:

- $n = 1$ dense: Site percolation
- $n = 1$ dilute: Ising model
- $n = 0$ dilute: Self-avoiding walks
- $n = 2$ either: Gaussian free field, XY model

Most of there are really *logarithmic* CFTs.

Our first objective is to understand the case of 'generic' n.

CFT of the $O(n)$ model [Di Francesco-Saleur-Zuber '87]

Central charge

$$c = 13 - 6\beta^2 - 6\beta^{-2} \qquad \text{with} \qquad \begin{cases} \Re\beta^2 > 0 \,, \\ \beta^2 \notin \mathbb{Q} \,. \end{cases}$$

Conformal weight Δ and momentum P:

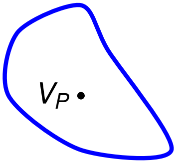
$$\Delta = P^2 - P^2_{(1,1)} \,, \quad \Delta_{(r,s)} = P^2_{(r,s)} - P^2_{(1,1)} \,, \quad P_{(r,s)} = \tfrac{1}{2}\left(-\beta r + \beta^{-1} s\right) \,.$$

Field content, with left- and right-moving conformal weights $(\Delta, \bar{\Delta})$:

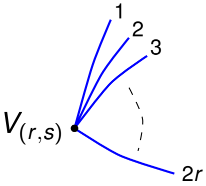
Name	Notation	Parameters	$(\Delta, \bar{\Delta})$
Degenerate	$V^d_{\langle r,s\rangle}$	$r = 1; s \in 2\mathbb{N}+1$	$\left(\Delta_{(r,s)}, \Delta_{(r,s)}\right)$
Diagonal	V_P	$P \in \mathbb{C}$	$\left(P^2 - P^2_{(1,1)}, P^2 - P^2_{(1,1)}\right)$
Non-diagonal	$V_{(r,s)}$	$r \in \frac{1}{2}\mathbb{N}^*; s \in \frac{1}{r}\mathbb{Z}$	$\left(\Delta_{(r,s)}, \Delta_{(-r,s)}\right)$

Interpretation of fields within the loop model:

Diagonal and non-diagonal fields



$w(P)$



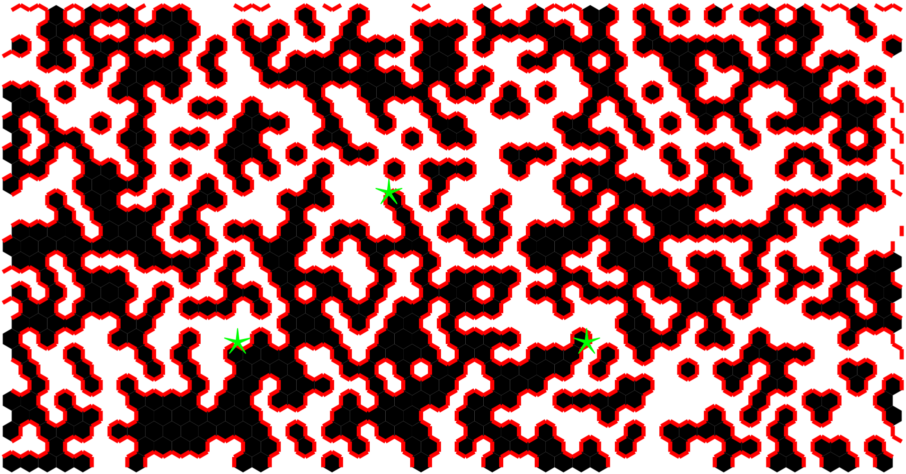
$\exp \frac{i}{2} s \sum_{k=1}^{2r} \theta_k$

$V^d_{\langle 1,3\rangle}$ is the energy operator.
The dense $O(n)$ model has a CFT limit iff $V^d_{\langle 1,3\rangle}$ is irrelevant:

$$\Re\Delta_{(1,3)} > 1 \iff \Re\beta^{-2} > 1 \ .$$

Correlation functions

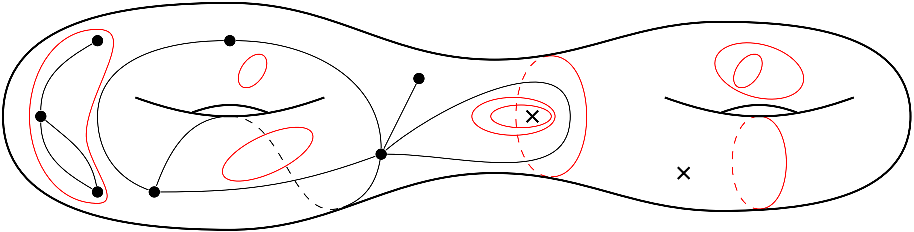
For example: Site percolation ($n = 1$, dense phase)



$P(z_1, z_2, z_3 \in \text{same hull}) = \frac{C_{2,2,2}}{|z_1-z_2|^{2\Delta}|z_2-z_3|^{2\Delta}|z_3-z_1|^{2\Delta}}$ by global conf. inv.

$\Delta = \frac{1}{8}$ and $C_{2,2,2} = ???$

Dream about correlation functions



Here $\bullet$ are $V_{(r,s)}$ insertions, and $\times$ are V_P insertions.

Open curves define a *combinatorial map* on a Riemann surface.

Segal's axioms: Three basic building blocks
1) Annulus with one insertion, 2) Disk with two insertions, 3) Pants.
The blocks are glued by integrating over eigenstates.

Progress thus far

- Fields related to irreps of $u\mathscr{J}\mathscr{T}\mathscr{L}_L(n)$.
- Bijection between correlation functions and combinatorial maps.
- Conformal symmetry enhanced to *interchiral symmetry* via $V^d_{\langle 1,3\rangle}$.
- Global $O(n)$ symmetry in interplay with conformal symmetry.

First goal is to understand $N \leq 4$ points on the sphere.

- $N = 2$ understood from critical exponents.
- $N = 3$ now understood in all cases.

  Degenerate operators [Dotsenko-Fateev '84,...].

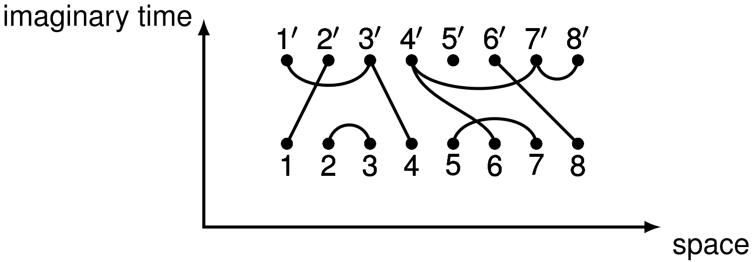
  Diagonal operators [Ikhlef-JJ-Saleur '16, Ang–Cai–Sun–Wu '24].

  Non-diagonal operators [Ang–Cai–Sun–Wu '24, JJ-Nivesvivat-Ribault-Roux '25].
- $N = 4$ from conformal bootstrap. Partial analytical control.

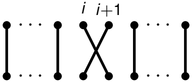
The talk focusses on $N = 3$ and non-diagonal operators.

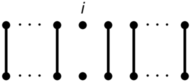
Diagrammatic algebras

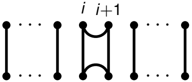
Partition algebra:



Generators:

$p_i =$ 

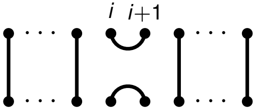
, $s_i =$ 

, $s_{i+\frac{1}{2}} =$ 

$1 \leq i \leq L-1$ $\quad$ $1 \leq i \leq L$ $\quad$ $1 \leq i \leq L-1$

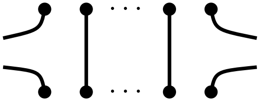
From this we can construct the TL generator:

$$e_i = s_{i+\frac{1}{2}} s_i s_{i+1} s_{i+\frac{1}{2}} =$$

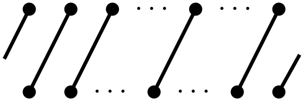


To get the periodic algebra $\mathscr{ATL}_L(n)$ we add two more generators:

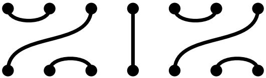
$$e_L =$$



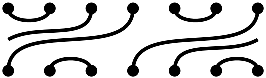
$$, \quad u =$$



Define also the pseudo-translation t of the $2r \in \mathbb{N}^*$ through-lines:



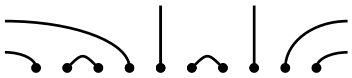
$$\xrightarrow{t}$$



$\mathscr{ATL}_L(n)$ is ∞-dimensional. A finite-dimensional quotient, the unoriented Jones–Temperley–Lieb algebra $u\mathscr{JTL}_L(n)$, is obtained by replacing non-contractible loops by n and imposing

$$t^{2r} \underset{u\mathscr{JTL}_L(n)}{=} 1 \ .$$

The standard modules $W^{(L)}_{(r,s)}$ are irreps of $u\mathscr{JTL}_L(n)$, spanned by link patterns with $2r$ defects. E.g. for $W^{(10)}_{(1,s)}$:



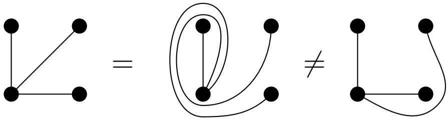
We have

$$\left(t - e^{\pi is}\right) W^{(L)}_{(r,s)} = 0 \ .$$

The labels (r, s) carry over to the CFT.

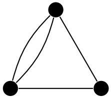
Combinatorial maps

A (connected) *combinatorial map* is a (connected) graph, together with a cyclic permutation of the half-edges around each vertex.
Monogons are forbidden.

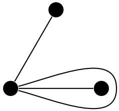


Case of $N = 3$ and non-diagonal operators:

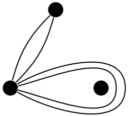
- For each choice of fields the combinatorial map is unique.
- If one field is diagonal, at least one pair of legs from a non-diagonal field must enclose it.



3, 3, 2 legs



4, 1, 1 legs



6, 2, 0 legs

Lessons from four-point functions

Two-point functions are given by the conformal dimensions, up to normalisation of the field.

Three-point functions are also fixed by global conformal invariance, up to structure constants.

Four-point functions could be determined by differential equations, if both $V^d_{\langle 1,s\rangle}$ and $V^d_{\langle r,1\rangle}$ were present, but we only have the former!

Therefore we need the *conformal bootstrap*.

But we can do better than usual for two reasons:

- $V^d_{\langle 1,3\rangle}$ generates an *interchiral symmetry*.
- We can exploit the global $O(n)$ symmetry.

Consider a four-point function of non-diagonal primary fields, and its s-channel decomposition into conformal blocks:

$$\left\langle \prod_{i=1}^{4} V_{(r_i,s_i)} \right\rangle = \sum_{s\in 2\mathbb{N}+1} D_s \mathscr{G}^D_{\langle 1,s\rangle} + \sum_{r\in\frac12\mathbb{N}^*} \sum_{s\in\frac1r\mathbb{Z}} D_{(r,s)} \mathscr{G}_{(r,s)} \ .$$

The blocks are known from Zamolodchikov's recursion relation.

Degenerate shift equations using $V^d_{\langle 1,3\rangle}$ determine $\frac{D_{(r,s+1)}}{D_{(r,s-1)}}$ and $\frac{D_{s+1}}{D_{s-1}}$.
So rewrite

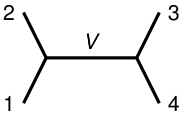
$$\left\langle \prod_{i=1}^{4} V_{(r_i,s_i)} \right\rangle = D_{s_0} \mathscr{H}_{s_0} + \sum_{r\in\frac12\mathbb{N}^*} \sum_{s\in\frac1r\mathbb{Z}\cap(-1,1]} D_{(r,s)} \mathscr{H}_{(r,s)} \ ,$$

in terms of interchiral blocks

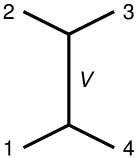
$$\mathscr{H}_{s_0} = \sum_{s\in s_0+2\mathbb{N}} \frac{D_s}{D_{s_0}} \mathscr{G}^D_{\langle 1,s\rangle} \quad , \quad \mathscr{H}_{(r,s)} = \sum_{j\in 2\mathbb{N}} \frac{D_{(r,s+j)}}{D_{(r,s)}} \mathscr{G}_{(r,s+j)} \ .$$

Solve then the crossing equations

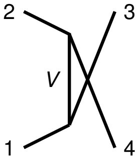
$$\sum_{V\in\mathcal{S}^{(s)}} D_V^{(s)}$$



$$= \sum_{V\in\mathcal{S}^{(t)}} D_V^{(t)}$$



$$= \sum_{V\in\mathcal{S}^{(u)}} D_V^{(u)}$$



s-channel *t*-channel *u*-channel

We know the spectrum. We can analyse the solutions in terms of the global $O(n)$ symmetry. The solution space can be constrained by a characteristic ("signature") of the desired combinatorial map.

In favourable cases this gives a unique (numerical) solution.

Conjecture: Each solutions to the crossing equations gives a valid correlation function in the $O(n)$ CFT.

We have computed the 30 correlation functions with $\sum_{i=1}^{4} r_i = 2, 3, 4$.

Digression on the Barnes double gamma function

Recall $c = 13 - 6\beta^2 - 6\beta^{-2}$ and set $Q = \beta + \beta^{-1}$.

For $\Re x > 0$ define $\Gamma_\beta(x)$ through

$$\log \Gamma_\beta(x) = \int_0^\infty \frac{\mathrm{d}t}{t} \left[\frac{\mathrm{e}^{-xt} - \mathrm{e}^{-Qt/2}}{(1-\mathrm{e}^{-\beta t})(1-\mathrm{e}^{-t/\beta})} - \frac{(Q/2-x)^2}{2\mathrm{e}^t} - \frac{Q/2-x}{t} \right]$$

and the shift equations

$$\frac{\Gamma_\beta(x+\beta)}{\Gamma_\beta(x)} = \sqrt{2\pi} \frac{\beta^{\beta x - \frac{1}{2}}}{\Gamma(\beta x)} \quad , \quad \frac{\Gamma_\beta(x+\beta^{-1})}{\Gamma_\beta(x)} = \sqrt{2\pi} \frac{\beta^{\frac{1}{2} - \beta^{-1} x}}{\Gamma(\beta^{-1} x)} \,.$$

Sometimes one defines also the upsilon function

$$\Upsilon_\beta(x) = \frac{1}{\Gamma_\beta(x)\Gamma_\beta(Q-x)} \,.$$

Reference 3-point structure constants

For non-diagonal fields, define:

$$C^{\text{ref}}_{(r_1,s_1)(r_2,s_2)(r_3,s_3)} = \prod_{\epsilon_1,\epsilon_2,\epsilon_3=\pm} \Gamma_\beta^{-1}\left(\tfrac{\beta+\beta^{-1}}{2} + \tfrac{\beta}{2}\left|\textstyle\sum_i \epsilon_i r_i\right| + \tfrac{\beta^{-1}}{2}\textstyle\sum_i \epsilon_i s_i\right) \ .$$

For diag fields, set $V_P = V_{(0,2\beta P)}$, so $C^{\text{ref}}_{(0,2\beta P_1)(0,2\beta P_2)(0,2\beta P_3)} = C_{P_1,P_2,P_3}$.

We observe that (up to normalisation by two-point functions):

- C_{P_1,P_2,P_3} equals the structure constants of diagonal fields [Ikhlef-JJ-Saleur '16, Ang–Cai–Sun–Wu '24]. (= imaginary DOZZ formula)
- The four-point amplitude in channel $x = s, t, u$ can be written

$$D^{(x)}_{(r,s)} = C^{\text{ref}}_{(r_1,s_1)(r_2,s_2)(r,s)} C^{\text{ref}}_{(r,s)(r_3,s_3)(r_4,s_4)} d^{(x)}_{(r,s)} \ ,$$

  where $d^{(x)}_{(r,s)}$ is a polynomial in the loop weights.
- $C_{2,2,2} = C^{\text{ref}}_{(1,0)(1,0)(1,0)}$ [Ang–Cai–Sun–Wu '24].

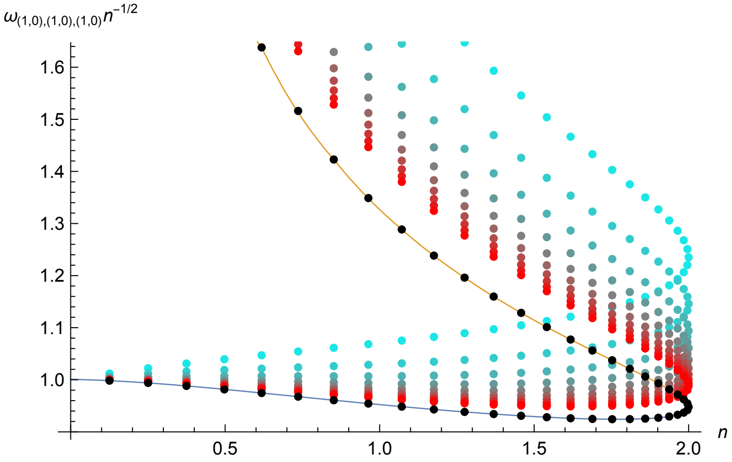
The conjecture

Three-point structure constants of combinatorial maps are simply given by $C^{\text{ref}}_{(r_1,s_1)(r_2,s_2)(r_3,s_3)}$, by taking a ratio that is invariant under field renormalizations $V_{(r,s)} \to \lambda_{(r,s)} V_{(r,s)}$:

$$\omega_{123} = C^{\text{ref}}_{123} \sqrt{\frac{C^{\text{ref}}_{000}}{C^{\text{ref}}_{011} C^{\text{ref}}_{022} C^{\text{ref}}_{033}}} ,$$

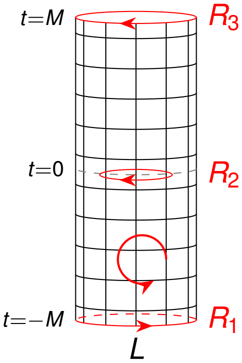
where 0 = Id = $V_{(0,1-\beta^2)}$.

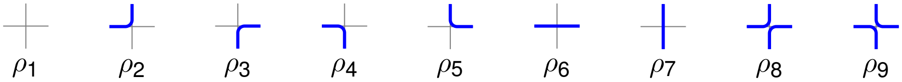
Numerical check for the case of 3 points $\in$ same loop:



This case is the one proved by [Ang–Cai–Sun–Wu '24].

Loop models on cylindrical lattices





$$Z = \sum_{\text{configurations}} n^{\#\text{loops}} \prod_{i=1}^{9} \rho_i^{V_i}$$

- The insertion of legs at the extremities (regions R_1 and R_3) is different from the middle (region R_2), so we normalise with care:

$$C_{123}(M,L) = \frac{Z_{123}}{Z_{220}} \sqrt{\frac{Z_{202} Z_{000}}{Z_{101} Z_{303}}}$$

- We show that the $M \to \infty$ limit of $C_{123}(M,L)$ is a ratio of form factors in $u\mathscr{JTL}_L(n)$ (up to unimportant signs), reading for the cases with (without) spins:

$$C_{123}(L) = \frac{\sum_{\sigma_2} e^{i\pi\sigma_2 s_2} \langle V_3 | \mathscr{O}_2^{\sigma_2} | V_1 \rangle}{\sum_{\sigma_2} e^{i\pi\sigma_2 s_2} \langle V_0 | \mathscr{O}_2^{\sigma_2} | V_2 \rangle} \qquad C_{123}(L) = \frac{\langle V_3 | \mathscr{O}_2 | V_1 \rangle}{\langle V_0 | \mathscr{O}_2 | V_2 \rangle}$$

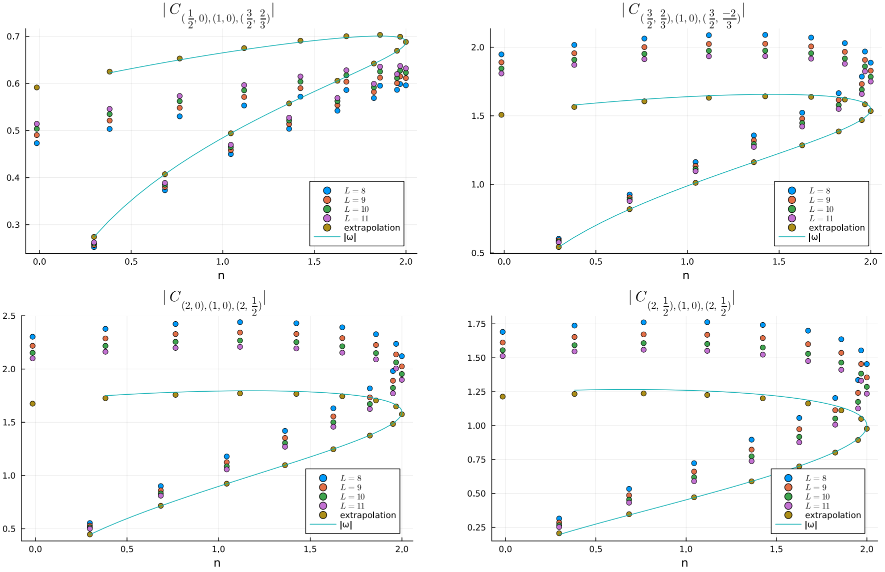
  The notation will be explained below.

- We then conjecture that

$$\lim_{L\to\infty} \lim_{M\to\infty} C_{123}(M,L) = \omega_{123}, \qquad s_1, s_2, s_3 \neq 1.$$

- To compute Z_{123} we use representations of $u\mathscr{JTL}_L(n)$ akin to the standard modules $W_{(r,s)}^{(L)}$, but with extra information.
- Transfer matrix T.
- The $\ell_i = 2r_i$ legs / defects are marked by their region of origin.
- Bottom and top legs carry representations $e^{i\pi s_1}$, $e^{i\pi s_3}$ of $\mathbb{Z}_{\ell_1}, \mathbb{Z}_{\ell_3}$.
- $|V_i\rangle = |V_{(r_i,s_i)}\rangle$ is dominant eigenvector of T (unique if $s_i \neq 1$).
- Middle operator $\mathscr{O}_{\ell_2}^{\sigma_2}$ inserts labelled legs on $r_2 = \ell_2/2$ broken edges.
- One then shows that $C_{123}(L)$ is independent of the eigenvalues of T and specifics of the boundary states in regions R_1, R_3.

A few representative tests



Conclusions

- An analytical prediction for all three-point structure constants in loop models.
  - Agrees with Coulomb gas for degenerate operators.
  - Agrees with imaginary DOZZ formula for diagonal operators.
  - But for non-diagonal (legged) operators, it goes beyond both frameworks.
- Precise link to results from the lattice model via a transfer-matrix construction.

Outstanding questions:

- Can our results be proved rigorously?
  - Uniform spanning trees (dense $n \to 0$ model) as a special case.
- Relate our $\mathcal{O}_2$ to lattice fusion [Ikhlef-Morin Duchesne '24-'25]?
- Can the necessary form factors be computed using integrability?
- How to derive four-point functions in loop models?