The planar parafermion algebra and N-state clock models

Murray Batchelor

International Workshop on "Challenges in Integrability"
9-21 Nov 2025, Wuhan, China

Australian National University

Outline of this talk

1) N-state clock models

   ◇ coupled Temperley-Lieb algebra

   ◇ pictorial presentations

2) Free parafermions

   ◇ exceptional points, non-Hermitian aspects

   ◇ non-Hermitian extension of the XY model

1) N-state clock models

- R. Adderton and MTB, *The planar parafermion algebra: The $Z(N)$ clock model and the coupled Temperley-Lieb algebra*, arXiv:2508.01672 (v2 coming soon)

- R. Adderton and MTB, *The planar parafermion algebra: Braid group representation*, in preparation

2) Exceptional points

- R.A. Henry and MTB, SciPost Physics **15**, 016 (2023)

- R.A. Henry, D.C. Liu and MTB, AAPPS Bull. **35**, 29 (2025)

Yang-Baxter Equations, Conformal Invariance and Integrability in Statistical Mechanics and Field Theory
Canberra July 10-14, 1989

Some Participants in the Workshop on Yang-Baxter Equations, Conformal Invariance and Integrability in Statistical Mechanics and Field Theory.

First (Front) Row: V. Pasquier, M. Jimbo, P.A. Pearce, R. Crewther, R.J. Baxter, G.R.W. Quispel, T. Miwa, H.J. de Vega.

Second Row: V. Jones, C.J. Hamer, A. Klümper, M.N. Barber, T.T. Truong, D.E. Evans, H. He, M.T. Batchelor, I.R. Aitchison.

Third Row: A.J. Bracken, P.J. Forrester, B.M. McCoy, J. Oitmaa, B. Nienhuis, B. Davies, M. Murray, A.L. Carey, V. Rittenberg, V. Dotsenko.

Fourth (Back) Row: R.B. Zhang, V. Korepin, C.A. Tracy, J.H.H. Perk, N. Joshi, K.A. Seaton, P. Jarvis, C.A. Hurst.

Some history

Conjecture: Murray has sat through more talks on the chiral Potts model than anyone else.

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Proof: He was at ANU in Canberra from the mid 80's.

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In 1989 Rodney gave a blackboard talk to a very small audience on his results for a certain new $Z(N)$ spin chain..

N-state clock models $\leftrightarrow$ $Z(N)$ spin chains

Building blocks are the $N \times N$ ('shift' and 'clock') matrices

$$
\begin{aligned}
(\tau)_{\ell m} &= \delta_{\ell, m+1} \pmod{N} \\
\sigma &= \mathrm{diag}\left(1, \omega, \omega^2, \ldots, \omega^{N-1}\right)
\end{aligned}
$$

with $\omega = \mathrm{e}^{2\pi i/N}$. For $N = 3$,

$$
\tau = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}.
$$

With 1 the identity, they satisfy

$$
\tau^N = \sigma^N = 1, \qquad \tau^\dagger = \tau^{N-1}, \qquad \sigma^\dagger = \sigma^{N-1},
$$

$$
\sigma\tau = \omega\tau\sigma.
$$

Some well known Yang-Baxter integrable N-state quantum spin chains are of the form

$$\boxed{H = -\sum_{j=1}^{L} \sum_{n=1}^{N-1} a_n \left(\lambda\, \tau_j^n + \sigma_j^n \sigma_{j+1}^{N-n} \right)}$$

$$\tau_j = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \tau \otimes 1 \otimes \cdots \otimes 1$$
$$\sigma_j = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma \otimes 1 \otimes \cdots \otimes 1$$

where 1, τ and σ are $N \times N$ matrices, τ and σ occur in position j.

special cases

- N-state quantum Potts model

$$\boxed{a_n = 1} \tag{1}$$

- Fateev-Zamolodchikov $Z(N)$ model

$$\boxed{a_n = \frac{1}{\sin(\pi n/N)}} \tag{2}$$

- N-state superintegrable chiral Potts model

$$\boxed{a_n = \frac{2}{1-\omega^{-n}}} \tag{3}$$

Each model reduces to the quantum Ising model for $N = 2$.

Models (1) and (2) are equivalent for $N = 3$.

Potts and Temperley-Lieb

Recall the N-state quantum Potts representation of the TL algebra

$$\begin{aligned} e_{2j-1} &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \tau_j^n & j &= 1, \ldots, L \\ e_{2j} &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left(\sigma_j \sigma_{j+1}^\dagger\right)^n & j &= 1, \ldots, L-1 \end{aligned}$$

with

$$\begin{aligned} e_j^2 &= \sqrt{N}\, e_j \\ e_j e_{j\pm 1} e_j &= e_j \\ e_j e_i &= e_i e_j \qquad |i-j| > 1 \end{aligned}$$

Potts model hamiltonian $H_{\mathrm{P}} = -\sum e_j$

Ongoing decades of fun with TL models

- mappings between models, faithful representations etc
- combinatorics
- stochastic processes
- loop models
- knot theory

Several known generalizations of TL algebra

Some examples

- multi-coloured TL algebra

  Grimm and Pearce (1992)
  Bisch and Jones (1997)

  aka k-colour Fuss-Catalan algebras

- 'blob' algebra

  Martin and Saleur (1994)

Power of the algebraic/pictorial approach

It is known that the TL algebra, along with the pictorial representation, can be used to derive the full eigenspectrum of the TL Hamiltonian, in that case via the Bethe Ansatz

- Levy (1990) (1991)
- Martin and Saleur (1994)
- de Gier and Pyatov (2004)
- Nepomechie and Pimenta (2016)

Superintegrable chiral Potts (SICP) chain

$$H_{\mathrm{CP}} = -\sum_{j=1}^{L}\sum_{n=1}^{N-1}\left(\lambda\,\alpha_n\,\tau_j^n + \overline{\alpha}_n\left(\sigma_j\sigma_{j+1}^\dagger\right)^n\right)$$

$$\alpha_n = \frac{\mathrm{e}^{\mathrm{i}(2n-N)\phi/N}}{\sin(\pi n/N)}, \qquad \overline{\alpha}_n = \frac{\mathrm{e}^{\mathrm{i}(2n-N)\overline{\phi}/N}}{\sin(\pi n/N)}$$

- The chiral Potts model has an R-matrix when

$$\lambda\cos\phi = \cos\overline{\phi}$$

- $\phi = \overline{\phi} = 0$ is the Fateev-Zamolodchikov model
- $\phi = \overline{\phi} = \frac{\pi}{2}$ is the SICP model
- H_{SICP} only solved for periodic bc's!

H_{SICP} can be written in terms of a coupled TL algebra!

[$N = 3$ case, J Fjelstad and T Månsson, JPA 45, 155208 (2012)]

For general N there are $N-1$ generators $e_j^{(k)}$ which satisfy

$$\begin{aligned}
\left(e_j^{(k)}\right)^2 &= Q\, e_j^{(k)} \\
e_j^{(k)} e_{j\pm1}^{(\ell)} e_j^{(k)} &= e_j^{(k)} \\
e_i^{(k)} e_j^{(\ell)} &= e_j^{(\ell)} e_i^{(k)} && |i-j| > 1 \\
e_j^{(k)} e_j^{(\ell)} = e_j^{(\ell)} e_j^{(k)} &= 0 && k \neq \ell
\end{aligned}$$

with $Q = \sqrt{N}$.

For $N = 2$ this reduces to the single TL generator e_j.

For $N = 3$ we label the generators by $e_j = e_j^{(1)}$ and $f_j = e_j^{(2)}$.

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Letters

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Letter

A coupled Temperley–Lieb algebra for the superintegrable chiral Potts chain

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Abstract

The Hamiltonian of the N-state superintegrable chiral Potts (SICP) model is written in terms of a coupled algebra defined by $N-1$ types of Temperley–Lieb generators. This generalises a previous result for $N=3$ obtained by Fjelstad and Månsson (2012 *J. Phys. A: Math. Theor.* **45** 155208). A pictorial representation of a related coupled algebra is given for the $N=3$ case

In general we can write

$$
\begin{aligned}
e_{2j-1}^{(k)} &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left(\omega^{k} \tau_j\right)^{n} && j = 1, \ldots, L \\
e_{2j}^{(k)} &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left(\omega^{k} \sigma_j \sigma_{j+1}^{\dagger}\right)^{n} && j = 1, \ldots, L-1
\end{aligned}
$$

for $k = 1, \ldots, N-1$. Here $\omega = e^{2\pi \mathrm{i}/N}$.

Then, for periodic bc's

$$
\boxed{H_{\mathrm{SICP}} = \frac{2}{\sqrt{N}} \sum_{j=1}^{L} \sum_{k=1}^{N-1} (N-k) \left(\lambda e_{2j-1}^{(k)} + e_{2j}^{(k)}\right) - (\lambda+1)(N-1)L}
$$

And for open bc's

$$
\begin{aligned}
H_{\mathrm{SICP}} \quad = \quad & -(N-1)(L(\lambda+1)-1) \\
& +\frac{2}{\sqrt{N}} \sum_{j=1}^{L} \sum_{k=1}^{N-1} \lambda(N-k) e_{2j-1}^{(k)} \\
& +\frac{2}{\sqrt{N}} \sum_{j=1}^{L-1} \sum_{k=1}^{N-1} (N-k) e_{2j}^{(k)}
\end{aligned}
$$

The generators $e_j^{(k)}$ also satisfy additional cubic relations.

For the $N = 3$ case

$$\begin{aligned}
f_j e_{j\pm1} e_j &= \pm\,\mathrm{i}\left(\omega^{\mp1} e_{j\pm1} e_j - f_{j\pm1} e_j\right) + \omega^{\pm1} e_j \\
&= \pm\,\mathrm{i}\left(\omega^{\mp1} f_j e_{j\pm1} - f_j f_{j\pm1}\right) + \omega^{\pm1} f_j \\
e_j e_{j\pm1} f_j &= \mp\,\mathrm{i}\left(\omega^{\pm1} e_{j\pm1} f_j - f_{j\pm1} f_j\right) + \omega^{\mp1} f_j \\
&= \mp\,\mathrm{i}\left(\omega^{\pm1} e_j e_{j\pm1} - e_j f_{j\pm1}\right) + \omega^{\mp1} e_j \\
f_j f_{j\pm1} e_j &= \omega^{\pm1} f_j e_{j\pm1} e_j \\
e_j f_{j\pm1} f_j &= \omega^{\mp1} e_j e_{j\pm1} f_j
\end{aligned}$$

For $N = 4$ with $e_j = e_j^{(1)}$, $f_j = e_j^{(2)}$ and $g_j = e_j^{(3)}$, a typical cubic relation is of the type

$$f_1 e_2 e_1 = \frac{1}{2}(1-\mathrm{i}) e_2 e_1 - \frac{1}{2}(1+\mathrm{i}) g_2 e_1 - \mathrm{i} f_2 e_1 + \mathrm{i}\, e_1 \,.$$

For general N

$$\begin{aligned} e_i^{(k)} e_i^{(l)} &= \delta_{k,l} \sqrt{N} e_i^{(k)} \\ e_i^{(k)} e_j^{(l)} &= e_j^{(l)} e_i^{(k)}, \quad |i-j| > 1 \\ e_i^{(k)} e_{i\pm 1}^{(l)} e_i^{(m)} &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \omega^{\mp(n-l)(k-m)} e_{i\pm 1}^{(n)} e_i^{(m)} \\ &= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \omega^{\pm(n-l)(k-m)} e_i^{(k)} e_{i\pm 1}^{(n)} \end{aligned}$$

A different way to write the $N = 3$ SICP hamiltonian

This originates from the staggered nature of the coupled operators e_j and f_j between odd and even sites:

$$H_{\mathrm{SICP}} = -\frac{2}{\sqrt{3}} \sum_{j=1}^{L} \left[\lambda \left(e_{2j-1} - f_{2j-1} \right) + \left(e_{2j} - f_{2j} \right) \right].$$

Using the definitions, with $(\omega - \omega^2)/\sqrt{3} = \mathrm{i}$, gives

$$\begin{aligned} e_{2j-1} - f_{2j-1} &= \mathrm{i} \left(\tau_j - \tau_j^2 \right) \\ e_{2j} - f_{2j} &= \mathrm{i} \left[\sigma_j \sigma_{j+1}^\dagger - (\sigma_j \sigma_{j+1}^\dagger)^2 \right] \end{aligned}$$

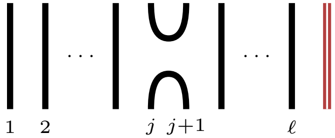
$$\boxed{H_{\mathrm{SICP}} = -\frac{2\,\mathrm{i}}{\sqrt{3}} \sum_{j=1}^{L} \left[\lambda \left(\tau_j - \tau_j^2 \right) + \sigma_j \sigma_{j+1}^\dagger - (\sigma_j \sigma_{j+1}^\dagger)^2 \right]}$$

The last terms $\sigma_L \sigma_{L+1}^\dagger$ and $(\sigma_L \sigma_{L+1}^\dagger)^2$ are omitted for open bc's.

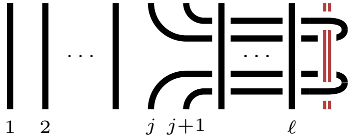
Pictorial presentation

We gave a pictorial presentation of the generators for $N = 3$:

$$e_j = e_j^{(1)} =$$



$$f_j = e_j^{(2)} =$$



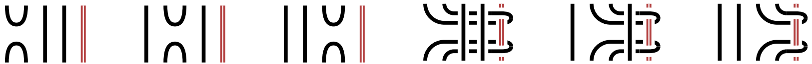
The key feature of the pictorial presentation is a pole around which loops can become entangled. Here we choose the position of the pole to be at one end of the chain. In the associated loop diagrams, closed (contractible) loops have weight Q, with $Q = \sqrt{3}$. The weight of closed (non-contractable) loops encircling the red line is zero.

SICP $N = 3$ example, generators e_j and f_j

The generators e_j are like the usual TL generators, with loops not encircling a line.

The generators f_j involve loops around the single red line.

Generators for the $L = 2$ site open chain:



e_1 e_2 e_3 f_1 f_2 f_3

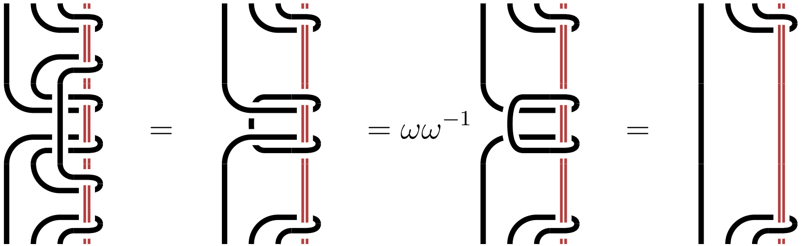
SICP $N = 3$ example, generators e_j and f_j

Algebraic relations can be proved via the diagrams.

We make use of the usual Kauffman-type relations.

The most interesting cubic relations are $f_1 f_2 f_1 = f_1$ and $f_2 f_1 f_2 = f_2$.

Proof of the relation $f_2 f_1 f_2 = f_2$



The knot can be resolved!

Key ingredients are crossing relations for loops encircling a red line.

Line crossing relations

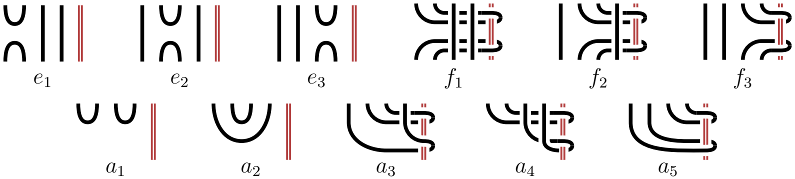
 $= \omega$  ,  $= \omega$ 

For this example with $N = 3$ the parameter is $\omega = \mathrm{e}^{\mathrm{i}2\pi/3}$.

This value can be derived topologically.

The open $N = 3$ SICP chain with $L = 2$

Generators and basis states for the $L = 2$ site open chain



Act on the basis states $a_1, \ldots, a_5$ with the generators and resolve the diagrams to construct eigenvalues of H_{loop} to match eigenvalues of H_{SICP}.

Concluding remarks for this part

We have given a pictorial presentation for the coupled algebra which defines the $N = 3$ state SICP model, for which the weight of contractable loops is $Q = \sqrt{3}$ and the weight of closed (non-contractable) loops encircling a line is zero.

There were/are many points to follow up!

Some examples:

- We have only discussed open boundary conditions.

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- Another known representation of this coupled algebra corresponds to the staggered spin-1/2 XX chain.
  It has two sets of generators e_i and f_i with $Q = 2$.

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- We have only discussed open boundary conditions.
- Another known representation of this coupled algebra corresponds to the staggered spin-1/2 XX chain.
  It has two sets of generators e_i and f_i with $Q = 2$.
- q-deformation?
- The algebraic approach opens up a path towards obtaining graphical based results for the SICP model.

But we found a problem with the diagrammatic proof for some of the algebraic commutation relations for the SICP model representation.

The diagrams for the $N = 3$ case give the correct cubic relations for a specific value of the parameter A in the Kauffmann bracket relation resolving the crossings but DO NOT give both cTL relations for far apart commutation, i.e., $f_1 f_3 = f_3 f_1$ and orthogonality $e_i f_i = 0$ for the same value of A.

There is no problem for the staggered XX diagrams.

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There is no problem for the staggered XX diagrams.

Fortunately there is a way to fix this.

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Communications in
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Planar Para Algebras, Reflection Positivity

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Published online: 22 December 2016 –

Abstract: We define a *planar para algebra*, which arises naturally from combining planar algebras with the idea of $\mathbb{Z}_N$ para symmetry in physics. A subfactor planar para algebra is a Hilbert space representation of planar tangles with parafermionic defects that are invariant under para isotopy. For each $\mathbb{Z}_N$, we construct a family of subfactor planar para algebras that play the role of Temperley–Lieb–Jones planar algebras. The first example in this family is the parafermion planar para algebra (PAPPA). Based on this example, we introduce parafermion Pauli matrices, quaternion relations, and braided relations for parafermion algebras, which one can use in the study of quantum information. An important ingredient in planar para algebra theory is the string Fourier transform (SFT), which we use on the matrix algebra generated by the Pauli matrices. Two different reflections play an important role in the theory of planar para algebras. One is the adjoint operator; the other is the modular conjugation in Tomita–Takesaki

Can show that the coupled Temperley-Lieb algebra admits a graphical presentation in terms of a 'planar' parafermion algebra PF_n defined by generators c_i, $i = 1, \ldots, n$ and relations

$$c_i^N = 1, \quad c_i^\dagger = (c_i)^{N-1}, \quad c_i c_j = \omega c_j c_i, \quad i < j.$$

$$c_{2j-1} = \Big(\prod_{k=1}^{j-1} \tau_k\Big)\sigma_j, \quad c_{2j} = \omega^{(N-1)/2} c_{2j-1}\tau_j$$

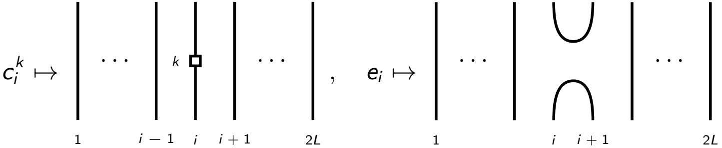
It follows that

$$\tau_i = \omega^{-(\frac{N-1}{2})} c_{2i-1}^\dagger c_{2i}, \quad \sigma_i \sigma_{i+1}^\dagger = \omega^{-(\frac{N-1}{2})} c_{2i} c_{2i+1}^\dagger$$

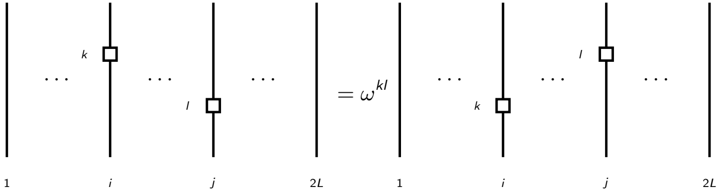
and so

$$e_{2i-1}^{(k)} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (\omega^{\frac{2k-N+1}{2}} c_{2i-1}^\dagger c_{2i})^n, \quad e_{2i}^{(k)} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (\omega^{\frac{2k-N+1}{2}} c_{2i} c_{2i+1}^\dagger)^n$$

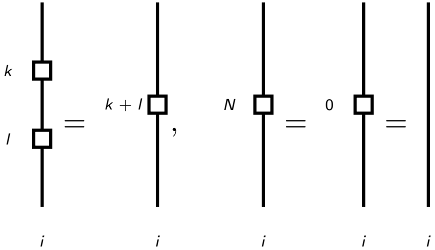
A natural diagrammatic presentation is given by denoting c_i^k as a k-labelled 1-box on the i-th strand. Let



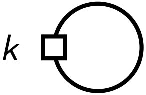
The k-labelled 1-box satisfies a ‘para-isotopy’ relation



And provides a representation of $\mathbb{Z}_N$ on a single strand:



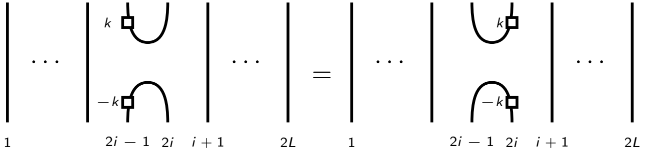
Contractable loops take value $\sqrt{N}$ and k-valued loops take value zero for $k \neq 0$:



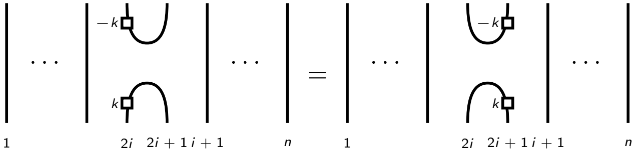
$$= \delta_{k,0}\sqrt{N} \mod N.$$

Define N copies of the Temperley-Lieb algebra as follows for $k \in \mathbb{Z}_N$

$e^{(k)}_{2i-1} \mapsto$



$e^{(k)}_{2i} \mapsto$



with $e^{(0)}_i = e_i$ as a standard cap-cup diagram.

All relations can be proved via this approach.

2) Free parafermions

Batchelor *et al. AAPPS Bulletin* (2023) 33:29
https://doi.org/10.1007/s43673-023-00105-3

AAPPS Bulletin

REVIEW ARTICLE **Open Access**

A brief history of free parafermions

Murray T. Batchelor[1*], Robert A. Henry[1] and Xilin Lu[1]

Abstract
In this article we outline the historical development and key results obtained to date for free parafermionic spin chains. The concept of free parafermions provides a natural N-state generalization of free fermions, which have long underpinned the exact solution and application of widely studied quantum spin chains and their classical counterparts. In particular, we discuss the Baxter-Fendley free parafermionic $Z(N)$ spin chain, which is a relatively simple non-Hermitian generalization of the Ising model.

Keywords Quantum spin chains, Free fermions, Free parafermions

1 Introduction

The concept of free fermions is fundamental to the celebrated exact solution of the two-dimensional Ising model in zero magnetic field [1–3] and its one-dimen- below, the Hamiltonian of this spin chain takes a similar form to that of the quantum Ising chain, but the spins have N allowed states, with Baxter's Hamiltonian reducing to the Ising model for $N = 2$. Baxter found that the

The Baxter-Fendley $Z(N)$ spin chain

A model that received **no attention** for a long time was found by Rodney Baxter in 1989.

R J Baxter, Phys Lett A **140**, 155 (1989); J Stat Phys **57**, 1 (1989)

For an L-site chain (with OBC) this model is defined as

$$\boxed{-H = \sum_{j=1}^{L} \tau_j + \lambda \sum_{j=1}^{L-1} \sigma_j^\dagger \sigma_{j+1}}$$

It reduces to the quantum Ising model for $N = 2$.

For $N = 3$ think of it as 'half' a Potts chain.

The eigenvalues of H have a simple form!

$$\boxed{-E = \omega^{p_1}\epsilon_1 + \omega^{p_2}\epsilon_2 + \cdots + \omega^{p_L}\epsilon_L}$$

for any choice of $p_k = 0, \ldots, N-1$. Recall $\omega = \mathrm{e}^{2\pi i/N}$.

- Initially a numerical observation.
- $N = 2$ is free fermions $\boxed{-E = \pm\epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_L}$
- Gives all N^L eigenvalues in the spectrum.
- The quasi-energies ϵ_k are known exactly.
- The model originates as the hamiltonian limit of the τ_2 model, a variant of the chiral Potts model.

- Fendley (2014) derived this result using a generalisation of the Jordan-Wigner transformation, namely the **Fradkin-Kadanoff transformation** to parafermionic operators originally introduced for the N-state clock models.

  P Fendley, J. Phys. A 47, 075001 (2014)

- Baxter (2014) and Au-Yang and Perk (2014,2016) applied Fendley's parafermionic approach to the more general τ_2 model with open boundaries.

  R J Baxter, J Phys A **47**, 315001 (2014)

  H Au-Yang and J H H Perk, J Phys A **47**, 315002 (2014); arXiv:1606.06319

- Fendley (2019) developed algebraic techniques which were also applied to multispin free fermion systems and adopted and generalised by Alcaraz and Pimenta (2020, 2021) to a class of multispin free fermion and free parafermion models.

The Baxter-Fendley hamiltonian is non-Hermitian, with complex energy eigenvalues for $N \geq 3$.

For any eigenvalue E, there are other eigenvalues $\omega E, \omega^2 E, \ldots$

This is the $Z(N)$ generalisation of the $E \leftrightarrow -E$ Ising symmetry (recall $\omega = -1$ for $N = 2$).

In general non-Hermitian hamiltonians describe the dynamics of physical systems that are not conservative.

The properties of the model are well worth exploring, being a rare example of an exactly solved non-Hermitian many-body system.

Boundary dependent bulk critical behaviour

F C Alcaraz and MTB, Phys. Rev. E 97, 062118 (2018)

• Solution of the free parafermionic $Z(N)$ chain is **only for open boundary conditions** (OBC).

• What about periodic boundary conditions (PBC)?

$\longrightarrow$ We resorted to a numerical study based on finite-size diagonalization and extrapolation.

bulk ground state energy per site

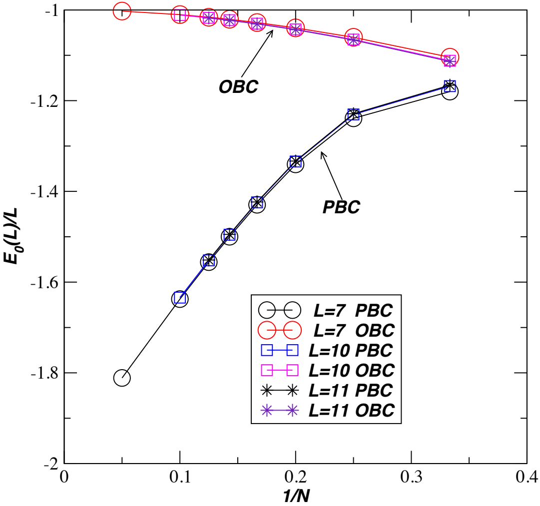
$$e_\infty(\lambda) = \lim_{L\to\infty} \frac{E_0(\lambda)}{L}$$

$N = 3$ $\lambda = 1$

	Extrap. PBC	Extrap. OBC	Exact OBC
$Z(3)$	-1.1544 $\pm$ 0.0002	-1.1321 $\pm$ 0.0002	-1.1320933607263

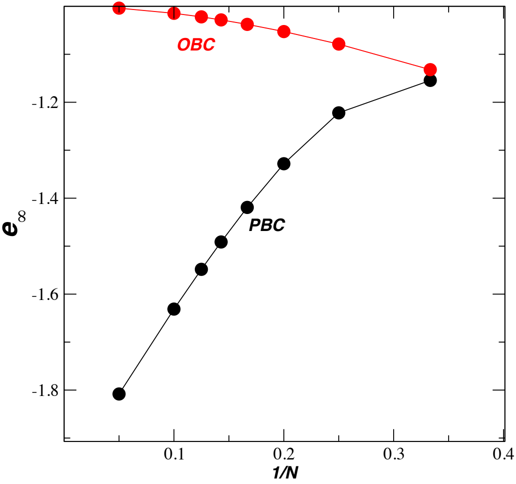
ground-state energy per site at $\lambda = 1$

data for $N = 3, 4, 5, 6, 7, 8, 10, 20$



ground-state energy per site at $\lambda = 1$

extrapolated data to bulk $L \to \infty$ limit



What are exceptional points?

Exceptional points are spectral singularities in the parameter space of a system in which two or more eigenvalues, and their corresponding eigenvectors, simultaneously coalesce.

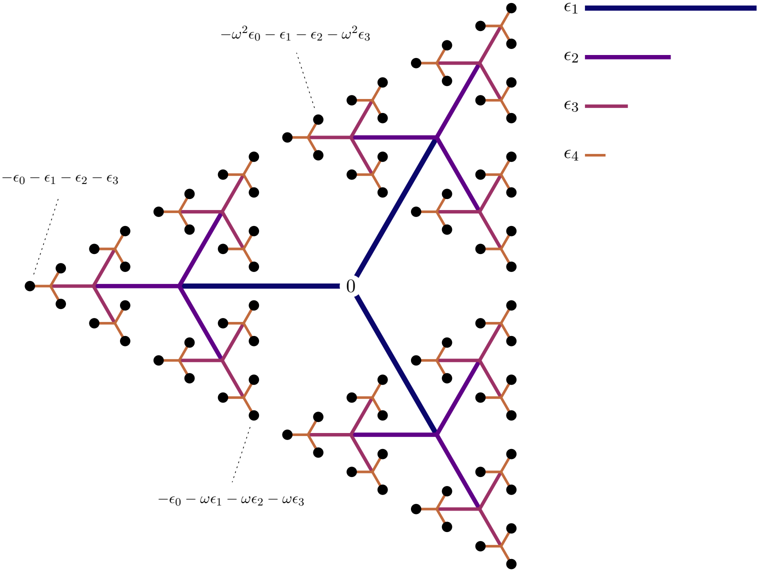
Such degeneracies are peculiar features of nonconservative systems that exchange energy with their surrounding environment.

EPs are level degeneracies induced by non-Hermiticity.

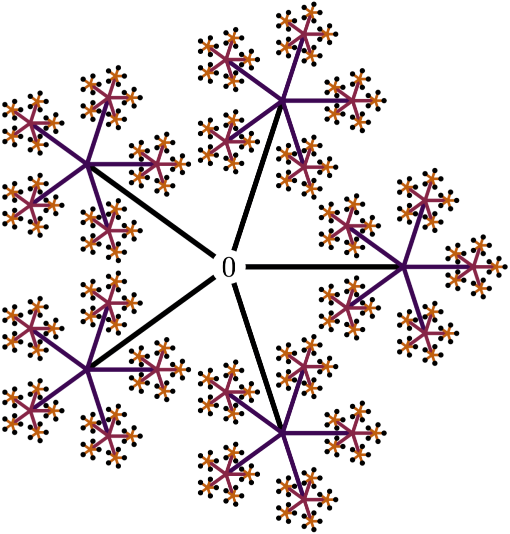
They exhibit exotic topological phenomena associated with the winding of eigenvalues and eigenvectors.

A vast and highly active topic!

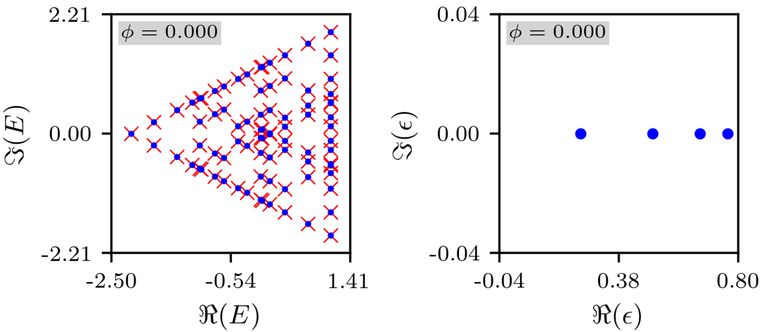
Free parafermion eigenspectrum $(N = 3,\ L = 4)$



$N = 5$



Free parafermion eigenspectrum ($N = 3,\ L = 4,\ \lambda = 1$)



(left) complex energy spectra obtained from the quasienergies (blue dots), and values obtained from exact diagonalisation of the full Hamiltonian (red crosses).

(right) quasienergies ϵ_j.

Partial motivation for the present work

Our previous work on this model showed unusual properties for $N \geq 3$ such as diverging correlation functions with system size L.

Z.-Z. Liu, R. A. Henry, MTB and H.-Q. Zhou, JSTAT 2019, 124002

Some ground-state expectation values for the free parafermion Z(N) spin chain

How to explain this behaviour?

The solution

F C Alcaraz, MTB and Z-Z Liu, J Phys A 50, 16LT03 (2017)

$$-H = \sum_{j=1}^{L} \tau_j + \lambda \sum_{j=1}^{L-1} \sigma_j \sigma_{j+1}^{\dagger}$$

$$-E = \sum_{j=1}^{L} \omega^{p_j} \epsilon_{k_j}, \quad p_j = 0, 1, \ldots, N-1, \quad \omega = \mathrm{e}^{2\pi i/N}$$

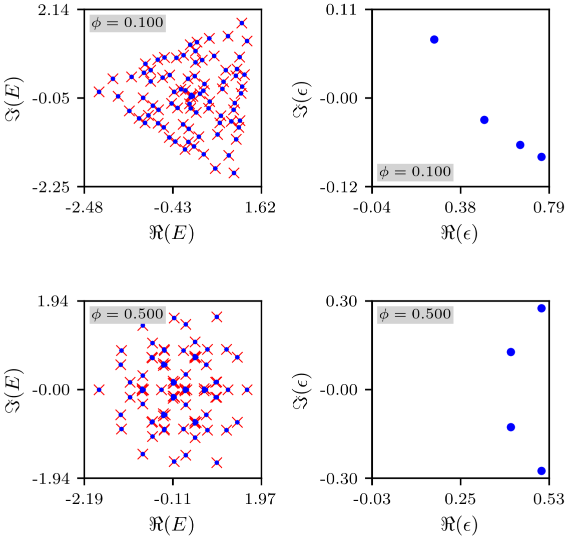
$$\boxed{\epsilon_k = \left(1 + \lambda^N + 2\lambda^{N/2} \cos k\right)^{1/N}}$$

k_j satisfy

$$\boxed{\sin(L+1)k = -\lambda^{N/2} \sin Lk}$$

for $\lambda = 1$, $k_j = \frac{2j\pi}{2L+1}$, $j = 1, \ldots, L$ and $\epsilon_k = \left(2 \cos \frac{k}{2}\right)^{2/N}$.

Free parafermion eigenspectrum ($N = 3$, $L = 4$, $\lambda = e^{2\pi i \phi/N}$)



Exceptional points

For real positive λ, the quasi-energies ϵ_j are always positive and distinct.

For **complex** λ, a pair of them may become equal at certain values of λ, which depend on N and L.

We call these *quasi-energy exceptional points*.

We call EPs in the energy spectrum *Hamiltonian exceptional points*.

$\Longrightarrow$ quasi-energy EPs give rise to Hamiltonian EPs.

Moreover, we can calculate them.

A quasi-energy EP will occur when

$$\sin(L+1)k = -\lambda^{N/2} \sin Lk$$

has a repeated root, meaning that both this equation and its derivative are satisfied.

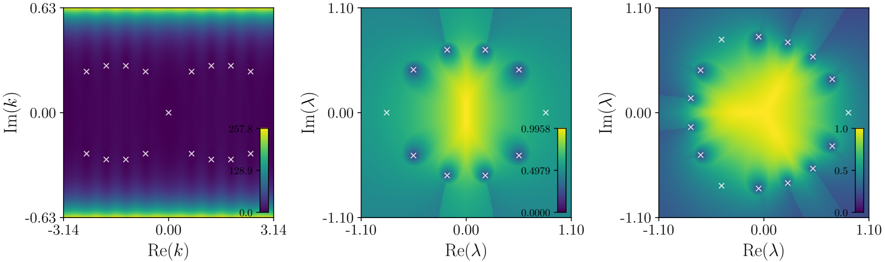
The EPs are pairs of values k_{EP} and λ_{EP} which satisfy these equations simultaneously.

In this way we obtain k_{EP} as the solution to

$$\sin(2L+1)k - (2L+1)\sin k = 0\,,$$

with the corresponding value λ_{EP} given by

$$\lambda = \left[\frac{-\sin(L+1)k}{\sin Lk}\right]^{2/N} .$$

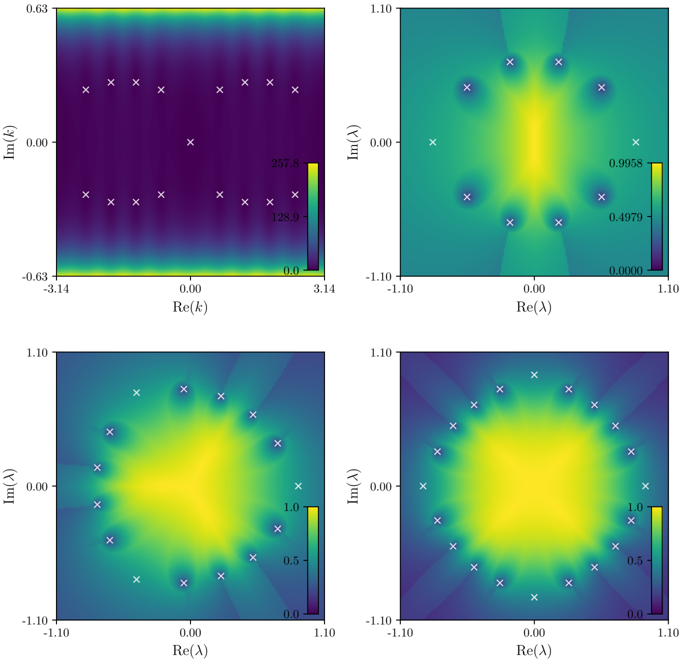


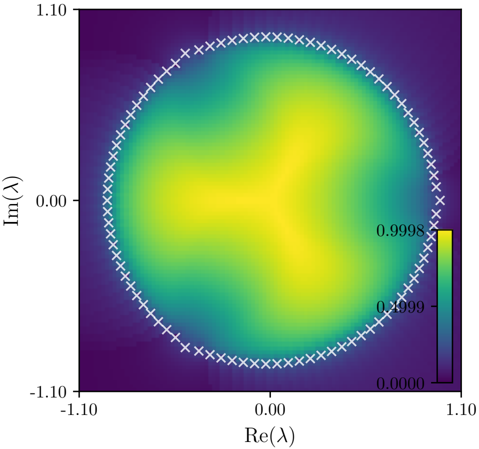
(left) k solutions for $L = 4$

(middle) difference between smallest and second-smallest quasi-energies for $N = 2$

(right) difference between smallest and second-smallest quasi-energies for $N = 3$

The corresponding values of λ_{EP} are also shown as crosses.

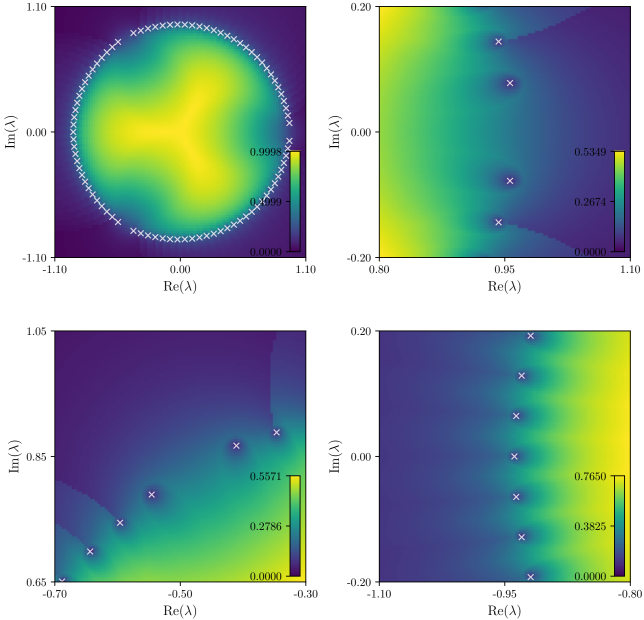

0.63
0.00
-0.63
-3.14
3.14
Im(k)
Re(k)
257.8
128.9
0.0
1.10
-1.10
Im(λ)
Re(λ)
0.9958
0.4979
0.0000
1.0
0.5



$N = 3 \quad L = 50$

Can apply large L expansion results for k to show that λ_{EP} satisfies

$$\lambda^N = \cos\left(\frac{2\pi j}{L}\right) \pm \mathrm{i}\sin\left(\frac{2\pi j}{L}\right) .$$


1.10
0.00
-1.10
Im(λ)
-1.10
0.00
1.10
Re(λ)
0.9998
0.4999
0.0000
0.20
0.00
-0.20
Im(λ)
0.80
0.95
1.10
Re(λ)
0.5349
0.2674
0.0000
1.05
0.85
0.65
Im(λ)
-0.70
-0.50
-0.30
Re(λ)
0.5571
0.2786
0.0000
0.20
0.00
-0.20
Im(λ)
-1.10
-0.95
-0.80
Re(λ)
0.7650
0.3825
0.0000

$N = 3 \quad L = 50$

Concluding remarks for this section

- We have located the quasi-energy EPs in the complex plane.
- Numerical tests confirm they correspond to Hamiltonian EPs.
- And also confirm that the corresponding eigenvectors coalesce.
- For large L they are on the unit circle in the complex λ plane.
- There are other degeneracies in the energy eigenspectrum, but they are not EPs.
- Although in the complex plane, EPs can influence properties (such as correlations) along the real axis..

details in R.A. Henry and MTB, SciPost Physics **15**, 016 (2023)

EPs in the XY model

The spin-$\frac{1}{2}$ anisotropic XY model is defined by the Hamiltonian

$$H = -\sum_{n=1}^{L-1} \left(\frac{1+\gamma}{2} \sigma_n^x \sigma_{n+1}^x + \frac{1-\gamma}{2} \sigma_n^y \sigma_{n+1}^y \right)$$

Free fermion eigenspectrum

$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_L$$

The quasienergies follow directly from the eigenvalues of the $L \times L$ matrix

$$\begin{pmatrix} y^2 & 0 & x\,y & 0 & \cdots & 0 \\ 0 & x^2+y^2 & 0 & x\,y & \cdots & 0 \\ x\,y & 0 & x^2+y^2 & 0 & \cdots & 0 \\ 0 & x\,y & 0 & x^2+y^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & x^2 \end{pmatrix}$$

where $x = \frac{1}{2}(1+\gamma)$ and $y = \frac{1}{2}(1-\gamma)$.

Solved in 1961 by Lieb, Schultz and Mattis.

For this model

$$\epsilon_j = \left[1 - (1 - \gamma^2) \sin^2 k_j\right]^{\frac{1}{2}} .$$

The k_j satisfy

$$\frac{\sin (L + 2)k}{\sin Lk} = \lambda^{\pm 1},$$

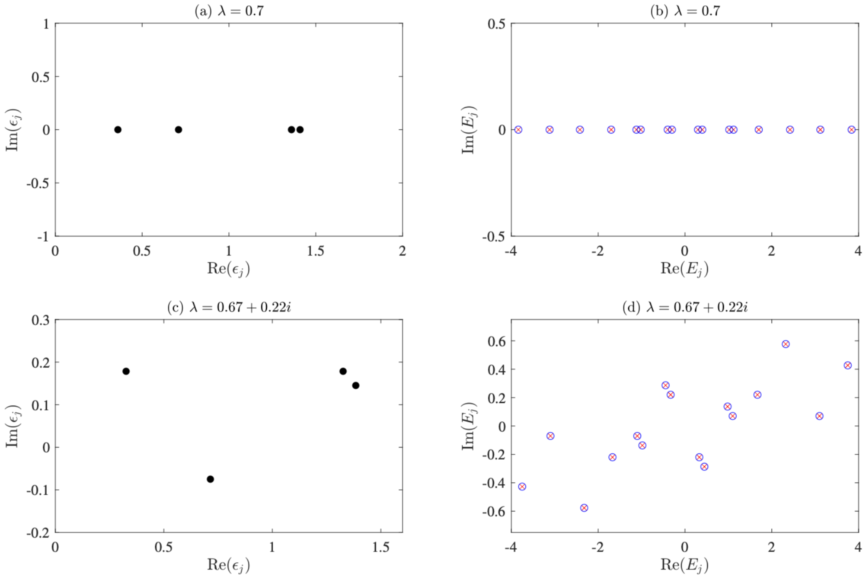
where

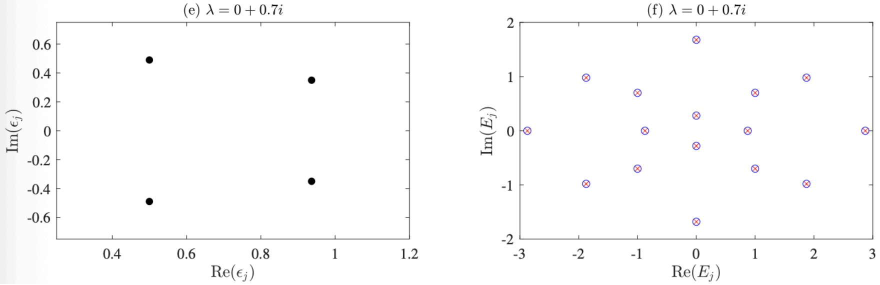
$$\lambda = \frac{1 - \gamma}{1 + \gamma} .$$

The solution also works for complex λ.

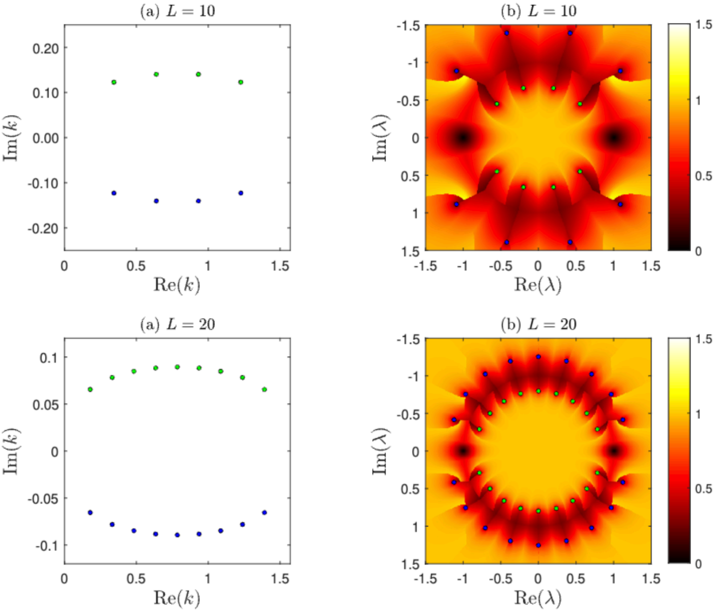
Work through in the same way to obtain the location of the exceptional points!

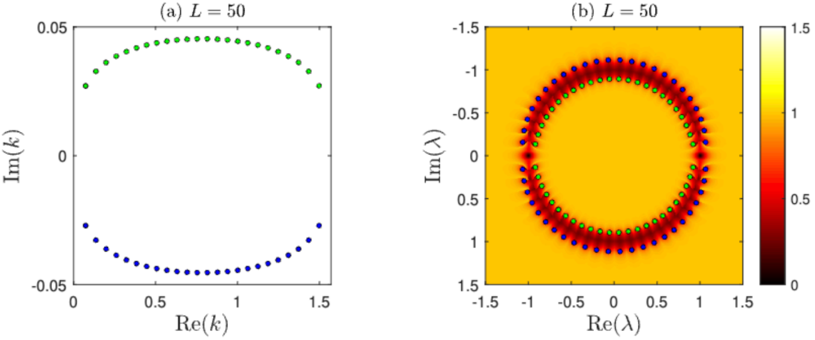
Examples of building the eigenspectrum ($L = 4$)





Can show (broken) PT-Symmetry if λ is pure imaginary.


(a) $L = 10$
(b) $L = 10$
(a) $L = 20$
(b) $L = 20$
Im(k)
Re(k)
Im(λ)
Re(λ)
0.20
0.10
0.00
-0.10
-0.20
0.1
0.05
0
-0.05
-0.1
0
0.5
1
1.5
-1.5
-1
-0.5



See two concentric rings of EPs which converge to the unit circle in the infinite size limit.

R.A. Henry, D.C. Liu and MTB, AAPPS Bull. 35, 29 (2025)

THANK YOU!