Root-TTbar Deformation in Integrable Sigma Models and 4D Duality-invariant Electrodynamics via Courant-Hilbert Approach

Bin Chen

Institute of Fundamental Physics and Quantum Technology,
Ningbo University
& School of Physics, Peking University

Challenges in Integrability, Wuhan, 9-21, November, 2025

H. Babaei-Aghbolagh, BC, and Song He, 2507.10137 and 2509.17075
BC, Jue Hou and Haowei Sun, 2405.04105

4D Nonlinear electrodynamics (NLED)

In terms of

$$S = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right), \quad P = -\frac{1}{4} F_{\mu\nu} \widetilde{F}^{\mu\nu} = \mathbf{E} \cdot \mathbf{B}.$$

The generic NLED Lagrangian density can be written as

$$\mathcal{L}(S, P)$$

Not all $\mathcal{L}(S, P)$ make sense!

4D Nonlinear electrodynamics (NLED)

In terms of

$$S = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right), \quad P = -\frac{1}{4} F_{\mu\nu} \widetilde{F}^{\mu\nu} = \mathbf{E} \cdot \mathbf{B}.$$

The generic NLED Lagrangian density can be written as

$$\mathcal{L}(S, P)$$

Not all $\mathcal{L}(S, P)$ make sense!

Basic requirements:
1) Weak-field limit: $\mathcal{L}(S, P) \to S + \mathcal{O}(S^2, P^2)$
2)Causality: EM radiation should not propagate faster than light.
Rule out all Langrangians of the form $\mathcal{L}(S)$ (except Maxwell) or $\mathcal{L}_1(S) + \mathcal{L}_2(P)$.

4D Nonlinear electrodynamics (NLED)

In terms of

$$S = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right), \quad P = -\frac{1}{4} F_{\mu\nu} \widetilde{F}^{\mu\nu} = \mathbf{E} \cdot \mathbf{B}.$$

The generic NLED Lagrangian density can be written as

$$\mathcal{L}(S, P)$$

Not all $\mathcal{L}(S, P)$ make sense!

Basic requirements:
1) Weak-field limit: $\mathcal{L}(S, P) \to S + \mathcal{O}(S^2, P^2)$
2)Causality: EM radiation should not propagate faster than light.
Rule out all Langrangians of the form $\mathcal{L}(S)$ (except Maxwell) or $\mathcal{L}_1(S) + \mathcal{L}_2(P)$.

Other possible requirements:
3) Analyticity: the weak field expansion does not contain square roots of S and P.
4)Electric-magnetic duality (in detail later)

Examples

1. Heisenberg-Euler Lagrangian in QED (1930s):

$$\mathcal{L}_{eff} = S + \frac{8\alpha^2}{45 m_e^2}\left(S^2 + \frac{7}{4}P^2\right) + \cdots$$

Examples

1. Heisenberg-Euler Lagrangian in QED (1930s):

$$\mathcal{L}_{eff} = S + \frac{8\alpha^2}{45 m_e^2}\left(S^2 + \frac{7}{4}P^2\right) + \cdots$$

2. From Born to Born-Infeld(1933-1934):

$$\begin{aligned}
\mathcal{L}_B &= -\sqrt{T^2 - 2TS} + T, \qquad T \text{ is a positive constant} \\
\mathcal{L}_{BI} &= -\sqrt{T^2 - 2TS - P^2} + T \\
&= -T\sqrt{-\det\left(\eta + F/\sqrt{T}\right)} + T
\end{aligned}$$

The BI action appeared in String theory, as LEEA of D-branes.

Examples

1. Heisenberg-Euler Lagrangian in QED (1930s):

$$\mathcal{L}_{eff} = S + \frac{8\alpha^2}{45 m_e^2}\left(S^2 + \frac{7}{4}P^2\right) + \cdots$$

2. From Born to Born-Infeld(1933-1934):

$$\begin{aligned}\mathcal{L}_B &= -\sqrt{T^2 - 2TS} + T, \qquad T \text{ is a positive constant}\\ \mathcal{L}_{BI} &= -\sqrt{T^2 - 2TS - P^2} + T\\ &= -T\sqrt{-\det\left(\eta + F/\sqrt{T}\right)} + T\end{aligned}$$

The BI action appeared in String theory, as LEEA of D-branes.

Schrödinger (1935) noticed that BI theory has EM-duality symmetry. The Hamiltonian

$$\mathcal{H} = \sqrt{T^2 + T(\mathbf{D}^2 + \mathbf{B}^2) + |\mathbf{D}\times\mathbf{B}|^2} - T, \qquad \mathbf{D} = \frac{\partial \mathcal{L}_{BI}}{\partial \mathbf{E}}$$

is invariant under $U(1)$ rotation:

$$(\mathbf{D} + i\mathbf{B}) \mapsto e^{i\theta}(\mathbf{D} + i\mathbf{B}).$$

For $\theta = \frac{\pi}{2}$, this is the usual EM-duality: $\mathbf{D} \to -\mathbf{B}, \mathbf{B} \to \mathbf{D}$.

Self-duality in general NLED

Dual field strength

$$\widetilde{\mathbf{G}}^{\mu\nu} \equiv \frac{\partial \mathcal{L}(\mathbf{F})}{\partial \mathbf{F}_{\mu\nu}}$$

The equations of motion and the Bianchi identity are respectively

$$\partial_\mu \widetilde{\mathbf{G}}^{\mu\nu} = 0, \qquad \partial_\mu \widetilde{\mathbf{F}}^{\mu\nu} = 0$$

Duality: $\widetilde{\mathbf{F}} \leftarrow \widetilde{\mathbf{G}}, \qquad \widetilde{\mathbf{G}} \leftarrow -\widetilde{\mathbf{G}}$.

Self-duality in general NLED

Dual field strength

$$\widetilde{\mathbf{G}}^{\mu\nu} \equiv \frac{\partial \mathcal{L}(\mathbf{F})}{\partial \mathbf{F}_{\mu\nu}}$$

The equations of motion and the Bianchi identity are respectively

$$\partial_\mu \widetilde{\mathbf{G}}^{\mu\nu} = 0, \qquad \partial_\mu \widetilde{\mathbf{F}}^{\mu\nu} = 0$$

Duality: $\widetilde{\mathbf{F}} \leftarrow \widetilde{\mathbf{G}}, \quad \widetilde{\mathbf{G}} \leftarrow -\widetilde{\mathbf{G}}$.

The invariance of the Hamiltonian $\mathcal{H}(\mathbf{D}, \mathbf{B})$ under $U(1)$ rotation

$$(\mathbf{D} + i\mathbf{B}) \mapsto e^{i\theta}(\mathbf{D} + i\mathbf{B}).$$

implies a differential equation for the Lagrangian

Gibbons and Rasheed (1995), Gaillard-Zumino 9712103

$$(\mathcal{L}_S)^2 - 2\,\frac{S}{P}\,\mathcal{L}_S\,\mathcal{L}_P - \left(\mathcal{L}_P\right)^2 = 1, \qquad \mathcal{L}_{S,P} \equiv \partial\mathcal{L}/\partial(S,P)$$

which can be recast into the form

$$\mathcal{L}_U\,\mathcal{L}_V = -1,$$

with

$$U = \frac{1}{2}(\sqrt{S^2 + P^2} - S), \quad V = \frac{1}{2}(\sqrt{S^2 + P^2} + S).$$

2D Integrable models

Two-dimensional integrable sigma models with group-valued fields $g(\sigma, \tau)$ are characterized by the flatness of a Lax connection. In light-cone coordinates, define the currents $\mathbf{j} = g^{-1}\mathbf{d}g, \quad j_\pm = j_0 \pm j_1$ and the scalar invariants

$$P_1 = -\mathrm{tr}[j_+ j_-], \quad P_2 = \tfrac{1}{2}\left(\mathrm{tr}[j_+ j_+]\,\mathrm{tr}[j_- j_-] + (\mathrm{tr}[j_+ j_-])^2\right).$$

The Lagrangian $\hat{\mathcal{L}}(P_1, P_2)$ must satisfy the PDE: R. Borsato et al. 2209.14274

$$8(P_1^2 - P_2)\,\hat{\mathcal{L}}_{P_2}^2 + 8P_1\,\hat{\mathcal{L}}_{P_2}\,\hat{\mathcal{L}}_{P_1} + 4\,\hat{\mathcal{L}}_{P_1}^2 = 1.$$

Introducing

$$q_1 = \frac{1}{4}(\sqrt{2P_2 - P_1^2} - P_1), \qquad q_2 = \frac{1}{4}(\sqrt{2P_2 - P_1^2} + P_1),$$

this reduces to a simple condition

$$\hat{\mathcal{L}}_{q_1}\,\hat{\mathcal{L}}_{q_2} = -1$$

which is exactly the same as the equation governing self-duality in 4D NLED.

Courant–Hilbert solution

$$\frac{\partial \mathcal{L}}{\partial U}\frac{\partial \mathcal{L}}{\partial V} = -1\,.$$

The general solution to this equation was first provided by Courant and Hilbert. The general solution is implicitly expressed in terms of (U, V) and a real function $\ell(\tau)$ of an auxiliary field τ as follows:

$$\mathcal{L} = \ell(\tau) - \frac{2U}{\dot{\ell}(\tau)}\,, \qquad \tau = V + \frac{U}{\dot{\ell}^2(\tau)}, \qquad \dot{\ell} = d\ell/d\tau.$$

At $U = 0$ and $\tau = V$, $\mathcal{L}(0, V) = \ell(V)$.
$\ell(\tau)$ is often referred to as the Courant-Hilbert (CH) function.

Choose preferred $\ell(\tau)$ satisfying $\ell(\tau) = \tau + \mathcal{O}(\tau^2)$,

$$\tau = V + \frac{U}{\dot{\ell}^2(\tau)} \;\Rightarrow\; \tau(U, V)$$

Courant–Hilbert solution

$$\frac{\partial \mathcal{L}}{\partial U}\frac{\partial \mathcal{L}}{\partial V} = -1\,.$$

The general solution to this equation was first provided by Courant and Hilbert. The general solution is implicitly expressed in terms of (U, V) and a real function $\ell(\tau)$ of an auxiliary field τ as follows:

$$\mathcal{L} = \ell(\tau) - \frac{2U}{\dot{\ell}(\tau)}\,, \qquad \tau = V + \frac{U}{\dot{\ell}^2(\tau)}, \qquad \dot{\ell} = d\ell/d\tau.$$

At $U = 0$ and $\tau = V$, $\mathcal{L}(0, V) = \ell(V)$.
$\ell(\tau)$ is often referred to as the Courant-Hilbert (CH) function.

Choose preferred $\ell(\tau)$ satisfying $\ell(\tau) = \tau + \mathcal{O}(\tau^2)$,

$$\tau = V + \frac{U}{\dot{\ell}^2(\tau)} \;\Rightarrow\; \tau(U, V) \;\Rightarrow\; \mathcal{L}(U, V).$$

Courant–Hilbert solution

$$\frac{\partial \mathcal{L}}{\partial U}\frac{\partial \mathcal{L}}{\partial V} = -1\,.$$

The general solution to this equation was first provided by Courant and Hilbert. The general solution is implicitly expressed in terms of (U, V) and a real function $\ell(\tau)$ of an auxiliary field τ as follows:

$$\mathcal{L} = \ell(\tau) - \frac{2U}{\dot{\ell}(\tau)}\,, \qquad \tau = V + \frac{U}{\dot{\ell}^2(\tau)}, \qquad \dot{\ell} = d\ell/d\tau.$$

At $U = 0$ and $\tau = V$, $\mathcal{L}(0, V) = \ell(V)$.
$\ell(\tau)$ is often referred to as the Courant-Hilbert (CH) function.

Choose preferred $\ell(\tau)$ satisfying $\ell(\tau) = \tau + \mathcal{O}(\tau^2)$,

$$\tau = V + \frac{U}{\dot{\ell}^2(\tau)} \;\Rightarrow\; \tau(U, V) \;\Rightarrow\; \mathcal{L}(U, V).$$

Example: $\ell(\tau) = \tau$

$\boxed{\text{4D Maxwell}}$ or $\boxed{\text{2D Principal chiral model}}$

Causality in 4D NLED

$$\boxed{\dot{\ell}(\tau) \geq 1, \quad \ddot{\ell}(\tau) \geq 0, \, \forall \tau \geq 0}$$

Born-Infeld: $\ell(\tau) = -\sqrt{T(T - 2\tau)}$, which is causal.

Auxiliary field method Russo and Townsend, 2505.08869

To obtain analytic NLED, one may introduce auxiliary field and make the ansatzz

$$\mathcal{L}(q_1, q_2) = -\frac{q_1}{y} + y\, q_2 - \Omega(y),$$

where $y = e^{\varphi}$ and φ is auxiliary field, and $\Omega(y)$ is a master potential.

1. The variation with respect to y yields

$$\frac{\partial \Omega(y)}{\partial y} = \frac{q_1}{y^2} + q_2 \;\Rightarrow\; y,$$

ensuring compatibility with the duality-invariant condition and the integrability condition.

2. **Born-Infeld**: $\Omega(y) = T\left(\frac{y+y^{-1}}{2} - 1\right)$.

Auxiliary field method Russo and Townsend, 2505.08869

To obtain analytic NLED, one may introduce auxiliary field and make the ansatzz

$$\mathcal{L}(q_1, q_2) = -\frac{q_1}{y} + y\, q_2 - \Omega(y),$$

where $y = e^{\varphi}$ and φ is auxiliary field, and $\Omega(y)$ is a master potential.

1. The variation with respect to y yields

$$\frac{\partial\Omega(y)}{\partial y} = \frac{q_1}{y^2} + q_2 \;\Rightarrow\; y,$$

ensuring compatibility with the duality-invariant condition and the integrability condition.

2. **Born-Infeld**: $\Omega(y) = T\left(\frac{y+y^{-1}}{2} - 1\right)$.
3. CH function:

$$\ell(\tau) = \Omega_\varphi - \Omega(\varphi), \qquad \dot{\ell}(\tau) = y, \qquad \tau = \Omega_\varphi / y$$

4. Causality guarantees a unique and consistent solution for the auxiliary field:

$$\varphi \geq 0, \qquad \Omega_{\varphi\varphi} > \Omega_\varphi.$$

Auxiliary field method Russo and Townsend, 2505.08869

To obtain analytic NLED, one may introduce auxiliary field and make the ansatzz

$$\mathcal{L}(q_1, q_2) = -\frac{q_1}{y} + y\,q_2 - \Omega(y),$$

where $y = e^{\varphi}$ and φ is auxiliary field, and $\Omega(y)$ is a master potential.

1. The variation with respect to y yields

$$\frac{\partial\Omega(y)}{\partial y} = \frac{q_1}{y^2} + q_2 \;\Rightarrow\; y,$$

ensuring compatibility with the duality-invariant condition and the integrability condition.

2. **Born-Infeld**: $\Omega(y) = T\left(\frac{y+y^{-1}}{2} - 1\right)$.
3. CH function:

$$\ell(\tau) = \Omega_\varphi - \Omega(\varphi), \qquad \dot{\ell}(\tau) = y, \qquad \tau = \Omega_\varphi / y$$

4. Causality guarantees a unique and consistent solution for the auxiliary field:

$$\varphi \geq 0, \qquad \Omega_{\varphi\varphi} > \Omega_\varphi.$$

5. Choosing $\Omega(y) = \Omega(-y) \Rightarrow$ analyticity.

TTbar flow

Another way to get 4D NLED or 2D integrable theories is through TTbar and TTbar-like flow.

TTbar flow

Another way to get 4D NLED or 2D integrable theories is through TTbar and TTbar-like flow. Let's first look at the TTbar operator, which was originally introduced in 2D Smirnov and Zamolodchikov, 1608.05499

$$\mathcal{O} = -\det(T^{\mu}{}_{\nu}) = \frac{1}{2}(T^{\mu\nu}T_{\mu\nu} - (T^{\mu}{}_{\mu})^2)$$

In 2D, with $z = x + iy, \bar{z} = x - iy$,

$$T \equiv T_{zz}, \qquad \overline{T} \equiv T_{\bar{z}\bar{z}}, \qquad \Theta \equiv T_{z\bar{z}}$$

then

$$\mathcal{O} = -4\left(T\overline{T} - \Theta^2\right)$$

TTbar flow

Another way to get 4D NLED or 2D integrable theories is through TTbar and TTbar-like flow. Let's first look at the TTbar operator, which was originally introduced in 2D Smirnov and Zamolodchikov, 1608.05499

$$\mathcal{O} = -\det(T^{\mu}{}_{\nu}) = \frac{1}{2}(T^{\mu\nu}T_{\mu\nu} - (T^{\mu}{}_{\mu})^2)$$

In 2D, with $z = x + iy, \bar{z} = x - iy$,

$$T \equiv T_{zz}, \qquad \overline{T} \equiv T_{\bar{z}\bar{z}}, \qquad \Theta \equiv T_{z\bar{z}}$$

then

$$\mathcal{O} = -4\left(T\overline{T} - \Theta^2\right)$$

One may define the same operator in other dimensions.

TTbar flow

Another way to get 4D NLED or 2D integrable theories is through TTbar and TTbar-like flow. Let's first look at the TTbar operator, which was originally introduced in 2D Smirnov and Zamolodchikov, 1608.05499

$$\mathcal{O} = -\det(T^{\mu}{}_{\nu}) = \frac{1}{2}(T^{\mu\nu}T_{\mu\nu} - (T^{\mu}{}_{\mu})^2)$$

In 2D, with $z = x + iy, \bar{z} = x - iy$,

$$T \equiv T_{zz}, \quad \overline{T} \equiv T_{\bar{z}\bar{z}}, \quad \Theta \equiv T_{z\bar{z}}$$

then

$$\mathcal{O} = -4\left(T\overline{T} - \Theta^2\right)$$

One may define the same operator in other dimensions.

The operator is irrelevant, and induce the so-called TTbar-flow:

$$S^{(\lambda+\delta\lambda)} = S^{(\lambda)} + \delta\lambda \int d^2x\sqrt{|g|}\mathcal{O}^{(\lambda)},$$

The flow keeps solvability, integrability, ...Conti *et al.*, 1803.04748, BC *et al.*, 2102.01470,......

$$\boxed{\text{BI theory=Max + TTbar flow}}$$

Root-TTbar flow and ModMax

Root TTbar: marginal deformation

$$\mathcal{R} = \sqrt{\frac{1}{2} T^{\mu\nu} T_{\mu\nu} - \frac{1}{4} (T^{\mu}{}_{\mu})^2}$$

Root-TTbar Flow:

$$\frac{d\mathcal{L}^{(\gamma)}}{d\gamma} = \mathcal{R}^{(\gamma)}, \qquad T^{(\gamma)}_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta \mathcal{L}^{(\gamma)}}{\delta g^{\mu\nu}}$$

It was first found that under this flow, Maxwell theory flows to ModMax (Modified Maxwell) theory Babaei-Aghbolagh *et al.*, 2202.11156

$$\mathcal{L}_{MM} = \cosh(\gamma) S + \sinh(\gamma) \sqrt{S^2 + P^2},$$

which is not only EM-duality invariant, but also conformal invariant. Bandos *et al.*, 2007.09092

$$\boxed{\text{ModMax theory=Max + Root TTbar}}$$

Root-TTbar flow and ModMax

Root TTbar: marginal deformation

$$\mathcal{R} = \sqrt{\frac{1}{2} T^{\mu\nu} T_{\mu\nu} - \frac{1}{4} (T^{\mu}{}_{\mu})^2}$$

Root-TTbar Flow:

$$\frac{d\mathcal{L}^{(\gamma)}}{d\gamma} = \mathcal{R}^{(\gamma)}, \qquad T^{(\gamma)}_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta \mathcal{L}^{(\gamma)}}{\delta g^{\mu\nu}}$$

It was first found that under this flow, Maxwell theory flows to ModMax (Modified Maxwell) theory Babaei-Aghbolagh *et al.*, 2202.11156

$$\mathcal{L}_{MM} = \cosh(\gamma) S + \sinh(\gamma) \sqrt{S^2 + P^2},$$

which is not only EM-duality invariant, but also conformal invariant. Bandos *et al.*, 2007.09092

$$\boxed{\text{ModMax theory} = \text{Max} + \text{Root TTbar}}$$

Root-TTbar was then widely studied in 2D models, holography, etc..

Babaei-Aghbolagh *et al.* 2206.12677, Conti *et al.*, 2206.03415, ...

In Courant-Hilbert formalism, the CH function

$$\ell(\tau) = e^{\gamma}\tau,$$

leads to ModMax theory in 2D &4D

$$\mathcal{L}_{MM} = -e^{-\gamma} q_1 + e^{\gamma} q_2, \qquad \tau = e^{-2\gamma} q_1 + q_2$$

In 4D

$$\mathcal{L}_{MM} = \cosh(\gamma) S + \sinh(\gamma)\sqrt{S^2 + P^2},$$

In 2D, it gives deformed PCM.

Flow & Courant-Hilbert

In a general d dimension

$$\begin{aligned}
\mathcal{R}^{(\gamma)} &= \frac{1}{\sqrt{d}}\sqrt{T_{\mu\nu}T^{\mu\nu} - \frac{1}{d}T_{\mu}{}^{\mu}T_{\nu}{}^{\nu}}, \\
\mathcal{O}^{(\lambda)} &= \frac{1}{d}\left(T_{\mu\nu}T^{\mu\nu} - \frac{2}{d}T_{\mu}{}^{\mu}T_{\nu}{}^{\nu}\right),
\end{aligned}$$

With a generic solution in terms of Courant-Hilbert function

$$\mathcal{R}^{(\gamma)} = \tau\ell'(\tau), \quad \mathcal{O}^{(\lambda)} = -\,\ell(\tau)(\ell(\tau) - 2\tau\ell'(\tau)).$$

A single Courant–Hilbert framework provides a unified, dimension-independent classification of both 4D duality-invariant NLED and their two-dimensional integrable sigma-model counterparts. The following discussions work in both 2D and 4D.

Flow & Courant-Hilbert

In a general d dimension

$$\begin{aligned}\mathcal{R}^{(\gamma)} &= \frac{1}{\sqrt{d}}\sqrt{T_{\mu\nu}T^{\mu\nu} - \frac{1}{d}T_\mu{}^\mu T_\nu{}^\nu},\\ \mathcal{O}^{(\lambda)} &= \frac{1}{d}\left(T_{\mu\nu}T^{\mu\nu} - \frac{2}{d}T_\mu{}^\mu T_\nu{}^\nu\right),\end{aligned}$$

With a generic solution in terms of Courant-Hilbert function

$$\mathcal{R}^{(\gamma)} = \tau\ell'(\tau), \quad \mathcal{O}^{(\lambda)} = -\ell(\tau)(\ell(\tau) - 2\tau\ell'(\tau)).$$

A single Courant–Hilbert framework provides a unified, dimension-independent classification of both 4D duality-invariant NLED and their two-dimensional integrable sigma-model counterparts. The following discussions work in both 2D and 4D.
With the auxiliary field,

$$\mathcal{R}^{(\gamma)} = y\Omega'(y)\,, \quad \mathcal{O}^{(\lambda)} = -\Omega^2(y) + y^2\Omega'^2(y)\,.$$

Motivation

$$\begin{tikzcd} S_{max} \arrow[r, "\mathcal{O}^{(\lambda)}"] \arrow[d, "\mathcal{R}^{(\gamma)}"'] & S_{BI}(\lambda) \arrow[d, "\mathcal{R}^{(\gamma)}"] \\ S_{MM}(\gamma) \arrow[r, "\mathcal{O}^{(\lambda)}"] & ? \end{tikzcd}$$

Strategy

The root-$T\bar{T}$ operator commutes with the irrelevant $T\bar{T}$ operator!
The root-$T\bar{T}$ flow keeps integrability as well! Borsato *et al.*, 2209.14274
It can be understood as particular transformations of the Lax connection that leave the zero-curvature condition invariant

Strategy

The root-$T\bar{T}$ operator commutes with the irrelevant $T\bar{T}$ operator!
The root-$T\bar{T}$ flow keeps integrability as well! Borsato *et al.*, 2209.14274
It can be understood as particular transformations of the Lax connection that leave the zero-curvature condition invariant

1. Extension from a single coupling constant λ to a two-parameter space (λ, γ) Borsato *et al.*, 2209.14274
2. Preservation of the Root-TTbar flow condition for all γ-coupled theories

The fundamental transformation rule takes the general form:

$$\mathcal{L}(\lambda) \to \tilde{\mathcal{L}}(\lambda, \gamma) = \mathcal{L}_{seed} + \lambda\, \mathcal{L}_1(\gamma) + \lambda^2\, \mathcal{L}_2(\gamma) + ...,$$

where $\mathcal{L}_n(\gamma)$ is carefully constrained to maintain compatibility with the root-$T\bar{T}$ operator. The Lagrangian transformation $\mathcal{L}(\lambda) \to \tilde{\mathcal{L}}(\lambda, \gamma)$ is systematically implemented through coordinated transformations of the λ dependent function, $\ell(\tau)$, potential, $\Omega(y)$, and the coupling λ.

Courant–Hilbert approach

The most general integrable deformations follow from a Courant–Hilbert generating function admitting the weak-field expansion BabaeiAghbolagh *et al.*, 2504.10361

$$\ell^{(\lambda)}(\tau) = \tau + \lambda\,\mathcal{O}(\tau^2) + \lambda^2\,\mathcal{O}(\tau^3) + \cdots .$$

The deformation goes beyond standard TTbar.

Courant–Hilbert approach

The most general integrable deformations follow from a Courant–Hilbert generating function admitting the weak-field expansion BabaeiAghbolagh *et al.*, 2504.10361

$$\ell^{(\lambda)}(\tau) = \tau + \lambda\,\mathcal{O}(\tau^2) + \lambda^2\,\mathcal{O}(\tau^3) + \cdots .$$

The deformation goes beyond standard TTbar.

The marginal γ-flow corresponds to the universal flow equations through rescalings of $\ell^{\lambda}(\tau)$ and the coupling λ, as:

$$\ell^{(\lambda,\gamma)}(\tau) = e^{\gamma}\,\ell^{(\bar{\lambda})}(\tau), \qquad \bar{\lambda} = e^{\gamma}\,\lambda,$$

defining a universal class of integrable theories where the deformation

$$\partial_\gamma \hat{\mathcal{L}}^{(\lambda,\gamma)} = \tau\ell'(\tau),$$

preserves integrability via consistent transformations of the Lax structure while maintaining the Root-TTbar operator structure.

Auxiliary field

Under the γ-flow, the master potential transforms as

$$\Omega(y) \to \tilde{\Omega}(\tilde{y}) = e^{\gamma}\,\Omega(e^{-\gamma}y)\,, \qquad \lambda \to e^{\gamma}\,\lambda\,,$$

where $\Omega(y)$ encodes the λ-dependence and $\tilde{\Omega}(\tilde{y})$ is the modified potential with two parameters (λ, γ).
Boundary condition: $\tilde{\Omega}(y, 0) = \Omega(y)$.

Auxiliary field

Under the γ-flow, the master potential transforms as

$$\Omega(y) \;\to\; \tilde{\Omega}(\tilde{y}) = e^{\gamma}\, \Omega(e^{-\gamma} y)\,, \qquad \lambda \to e^{\gamma}\, \lambda\,,$$

where $\Omega(y)$ encodes the λ-dependence and $\tilde{\Omega}(\tilde{y})$ is the modified potential with two parameters (λ, γ).
Boundary condition: $\tilde{\Omega}(y, 0) = \Omega(y)$.

The general modified Lagrangian with two coupling constants (λ, γ) can be written in the form:

$$\tilde{\mathcal{L}}(q_1, q_2, \tilde{y}) = -\frac{q_1}{\tilde{y}} + \tilde{y}\, q_2 - \tilde{\Omega}(\tilde{y})\,.$$

Generalized Born-Infeld theoreis

Taking the CH-function

$$\ell^{(\lambda)}(\tau) = -\frac{1}{\lambda}\left(1 - \sqrt{1 + 2\lambda\tau}\right).$$

By applying the transformations $\ell^{(\lambda,\gamma)}(\tau) = e^{\gamma}\,\ell^{(\bar{\lambda})}(\tau)$, we find that

$$\hat{\mathcal{L}}_{GBI} = \frac{1}{\lambda}\left(-1 + \sqrt{1 - 2e^{-\gamma}\lambda q_1}\sqrt{1 + 2e^{\gamma}\lambda q_2}\right).$$

$$\begin{tikzcd}
S_{max} \arrow[r, "\mathscr{O}^{(\lambda)}"] \arrow[d, "\mathscr{R}^{(\gamma)}"'] & S_{BI}(\lambda) \arrow[d, "\mathscr{R}^{(\gamma)}"] \\
S_{MM}(\gamma) \arrow[r, "\mathscr{O}^{(\lambda)}"] & S_{GBI}(\lambda,\gamma)
\end{tikzcd}$$

Logarithmic theories

$$\ell(\tau) = -\frac{1}{\lambda}\log(1-\lambda\tau)\,.$$

$$\begin{aligned}\hat{\mathcal{L}}_{Log} &= \frac{1}{\lambda}\Bigg(-\sqrt{1+4\lambda e^{-\gamma}q_1-4\lambda^2 q_1 q_2}+1\\ &- \log\bigg(\frac{e^{\gamma}}{2\lambda q_1}\Big(1-\sqrt{1+4\lambda e^{-\gamma}q_1-4\lambda^2 q_1 q_2}\Big)\bigg)\Bigg).\end{aligned}$$

q-Deformed Theories

$$\ell(\tau) = \frac{1}{\lambda}\left(1 - \left(1 - \tfrac{1}{\mathbf{q}}\lambda\tau\right)^{\mathbf{q}}\right).$$

Let's focus on $q = 3/4$.

In the auxiliary field formalism, consider the potential

$$\Omega(y) = \frac{1}{4\lambda}\Big((y^{-3} + 3y) - 4\Big),$$

where y is determined by:

$$y = \sqrt{\frac{2\lambda q_1 + \sqrt{9 + 4\lambda(\lambda q_1^2 - 3q_2)}}{3 - 4\lambda q_2}}.$$

It leads to the Lagrangian

$$\begin{aligned}\mathcal{L}_{q=\frac{3}{4}} &= \frac{1}{\lambda} - 2q_1\big(1 - \frac{4\lambda}{9}(\Delta - 2\lambda q_1^2 + 3q_2)\big)^{1/4} \\ &- \frac{1}{\lambda}\big(1 - \frac{4\lambda}{9}(\Delta - 2\lambda q_1^2 + 3q_2)\big)^{3/4},\end{aligned}$$

with Δ defined as:

$$\Delta = \sqrt{q_1^2\big(9 + 4\lambda^2 q_1^2 - 3\lambda q_2)\big)}.$$

New integrable theories

$$\Omega(y) = \frac{1}{\lambda}\Big(\frac{2}{3y} + \tfrac{1}{3}y^2 - 1\Big), \qquad y = -\frac{\lambda}{2}(\Lambda^{\frac{1}{3}} - q_2 + \frac{q_2^2}{\Lambda^{\frac{1}{3}}}),$$

where Λ is given by:

$$\Lambda = \frac{1}{\lambda^3}\Big(2\sqrt{2 + 3\lambda q_1)(2 + 3\lambda q_1 + \lambda^3 q_2^3)} - 4 - 6\lambda q_1 - \lambda^3 q_2^3\Big).$$

The Lagrangian for this new theory takes the form:

$$\hat{\mathcal{L}}_{new} = \frac{1}{\lambda}\Big(1 - \frac{2q_1}{q_2 - \Lambda^{\frac{1}{3}} - \frac{q_2^2}{\Lambda^{\frac{1}{3}}}} + \frac{\lambda^2}{2} q_2(q_2 - \Lambda^{\frac{1}{3}} - \frac{q_2^2}{\Lambda^{\frac{1}{3}}})$$
$$- \quad \frac{2}{3\lambda(q_2 - \Lambda^{\frac{1}{3}} - \frac{q_2^2}{\Lambda^{\frac{1}{3}}})}\big(2 + \frac{1}{8}\lambda^3(q_2 - \Lambda^{\frac{1}{3}} - \frac{q_2^2}{\Lambda^{\frac{1}{3}}})^3)\Big).$$

Summary

4D self-dual NLED and 2D integrable models can be governed by one equation

$$\frac{\partial \mathcal{L}}{\partial q_1}\frac{\partial \mathcal{L}}{\partial q_2} = -1.$$

Summary

4D self-dual NLED and 2D integrable models can be governed by one equation

$$\frac{\partial \mathcal{L}}{\partial q_1}\frac{\partial \mathcal{L}}{\partial q_2} = -1.$$

The general solution to this equation is provided by Courant-Hilbert function $\ell(\tau)$, restricted by the causality condition.

Alternatively, one may solve the equation by introducing auxiliary field.

Summary

4D self-dual NLED and 2D integrable models can be governed by one equation

$$\frac{\partial \mathcal{L}}{\partial q_1}\frac{\partial \mathcal{L}}{\partial q_2} = -1.$$

The general solution to this equation is provided by Courant-Hilbert function $\ell(\tau)$, restricted by the causality condition.

Alternatively, one may solve the equation by introducing auxiliary field.

In this work, we further considered the Root-TTbar flow characterized by γ and construct two-parameter families of theories. We found new integrable theories in 2D: Generalized Born-Infeld; q-deformed; Logarithmic theories, ...

Discussions

Another method to solve the equation: Characteristic method BC *et al.*, 2506.03129

Discussions

Another method to solve the equation: Characteristic method BC *et al.*, 2506.03129

Carrollian NLED? Different duality transformation, similar Gaillard-Zumino condition, ...
E.g. Carrollian ModMax BC *et al.*, 2405.04105

$$\mathcal{L}_{\gamma}^{\mathsf{CMM}}(\mathcal{S},\mathcal{P}) = \frac{1}{4}\left(e^{\gamma}\mathcal{S} \mp e^{-\gamma}\frac{\mathcal{P}^2}{\mathcal{S}}\right),$$

with

$$\mathcal{S} \equiv \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = m^{\mu\rho}\gamma^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}, \quad \mathcal{P} \equiv \frac{1}{2}F_{\mu\nu}\widetilde{F}^{\mu\nu}.$$

Discussions

Another method to solve the equation: Characteristic method BC *et al.*, 2506.03129

Carrollian NLED? Different duality transformation, similar Gaillard-Zumino condition, ...
E.g. Carrollian ModMax BC *et al.*, 2405.04105

$$\mathcal{L}_{\gamma}^{\text{CMM}}(\mathcal{S},\mathcal{P}) = \frac{1}{4}\left(e^{\gamma}\mathcal{S} \mp e^{-\gamma}\frac{\mathcal{P}^2}{\mathcal{S}}\right),$$

with

$$\mathcal{S} \equiv \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = m^{\mu\rho}\gamma^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}, \quad \mathcal{P} \equiv \frac{1}{2}F_{\mu\nu}\widetilde{F}^{\mu\nu}.$$

The Carrollian ModMax theories in the family deform among themselves under Root-TTbar flow, except two end-points $\gamma \to \pm\infty$, where the flow is non-invertible.

$$\mathcal{L} \sim -\frac{\mathcal{P}^2}{2\mathcal{S}} \qquad\qquad \mathcal{L} \sim \frac{\mathcal{S}}{2}$$

$$-\infty \;\xleftrightarrow[\sqrt{T\bar{T}}-\gamma \text{ flow}]{\text{Carrollian ModMax}}\; +\infty$$

Thanks for your attention!