(Exact)Strong Zero Modes, and the XYZ chain without elliptic functions

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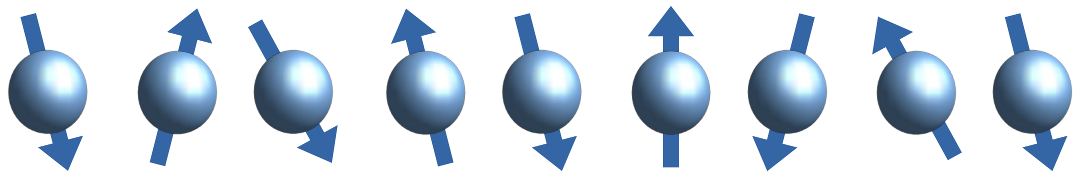
Based on :

- EV Yeh Mitra Piroli, *Strong Zero Modes in Integrable Quantum Circuits,* PRL (2024)
- Fendley Ghermann Verstraete EV, *XYZ integrability the easy way*, arXiv:2511.04674
- Essler Fendley EV, *Strong zero modes in open integrable spin-S chains*
- EV Fendley, in preparation

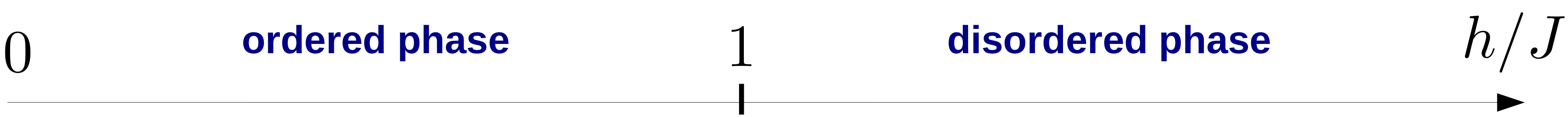
1. Boundary strong zero modes
2. Exact strong zero modes from integrability
3. XYZ revisited

Ising/Majorana chain

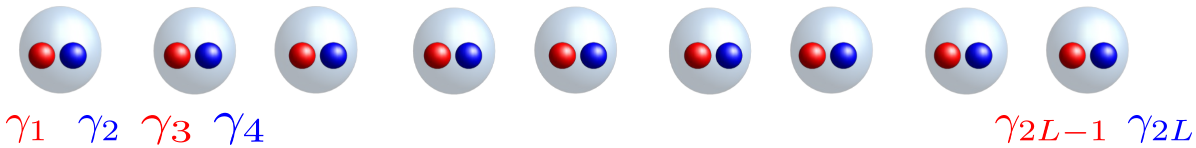


$$H = -h \sum_{j=1}^{L} \sigma_j^x - J \sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z$$

Z_2 symmetry : $$[\mathcal{Z}, H] = 0 \ , \quad \mathcal{Z} = \prod_{j=1}^{L} \sigma_j^x$$



Ising/Majorana chain



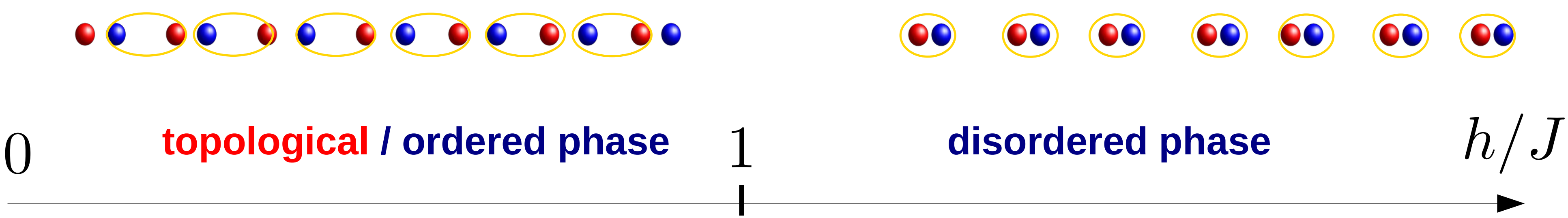
$$\gamma_{2j-1} = (\prod_{k<j} \sigma_k^x)\sigma_j^z$$

$$\gamma_{2j} = (\prod_{k<j} \sigma_k^x)\sigma_j^y$$

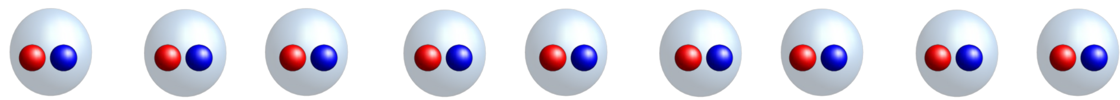
$$H = -h\sum_{j=1}^{L} \gamma_{2j-1}\gamma_{2j} - J\sum_{j=1}^{L-1} \gamma_{2j}\gamma_{2j+1}$$

Kitaev '01

Z_2 symmetry : $[\mathcal{Z}, H] = 0$, $\mathcal{Z} = \prod_{j=1}^{L} \sigma_j^x = (-1)^F$



Ising/Majorana chain



γ_1 γ_2 γ_3 γ_4 γ_{2L-1} γ_{2L}

$$\gamma_{2j-1} = \Big(\prod_{k<j} \sigma_k^x\Big)\sigma_j^z$$

$$\gamma_{2j} = \Big(\prod_{k<j} \sigma_k^x\Big)\sigma_j^y$$

$$H = -h\sum_{j=1}^{L} \gamma_{2j-1}\gamma_{2j} - J\sum_{j=1}^{L-1}\gamma_{2j}\gamma_{2j+1}$$

Kitaev '01

Z_2 symmetry : $[\mathcal{Z}, H] = 0$, $\quad \mathcal{Z} = \prod_{j=1}^{L} \sigma_j^x = (-1)^F$

Decoupled edge modes at *h=0*, which **persist throughout the ordered phase !**

0 **topological / ordered phase** 1 **disordered phase** h/J

Strong Zero Modes

Fendley '12

- At h=0, $[\sigma_1^z, H] = 0$
- For h<J, $[\Psi, H] = 0$ $\quad \Psi = \sigma_1^z + \frac{h}{J}\sigma_1^x\sigma_2^z + \left(\frac{h}{J}\right)^2 \sigma_1^x\sigma_2^x\sigma_3^z + \ldots$

up to corrections exponentially small in L

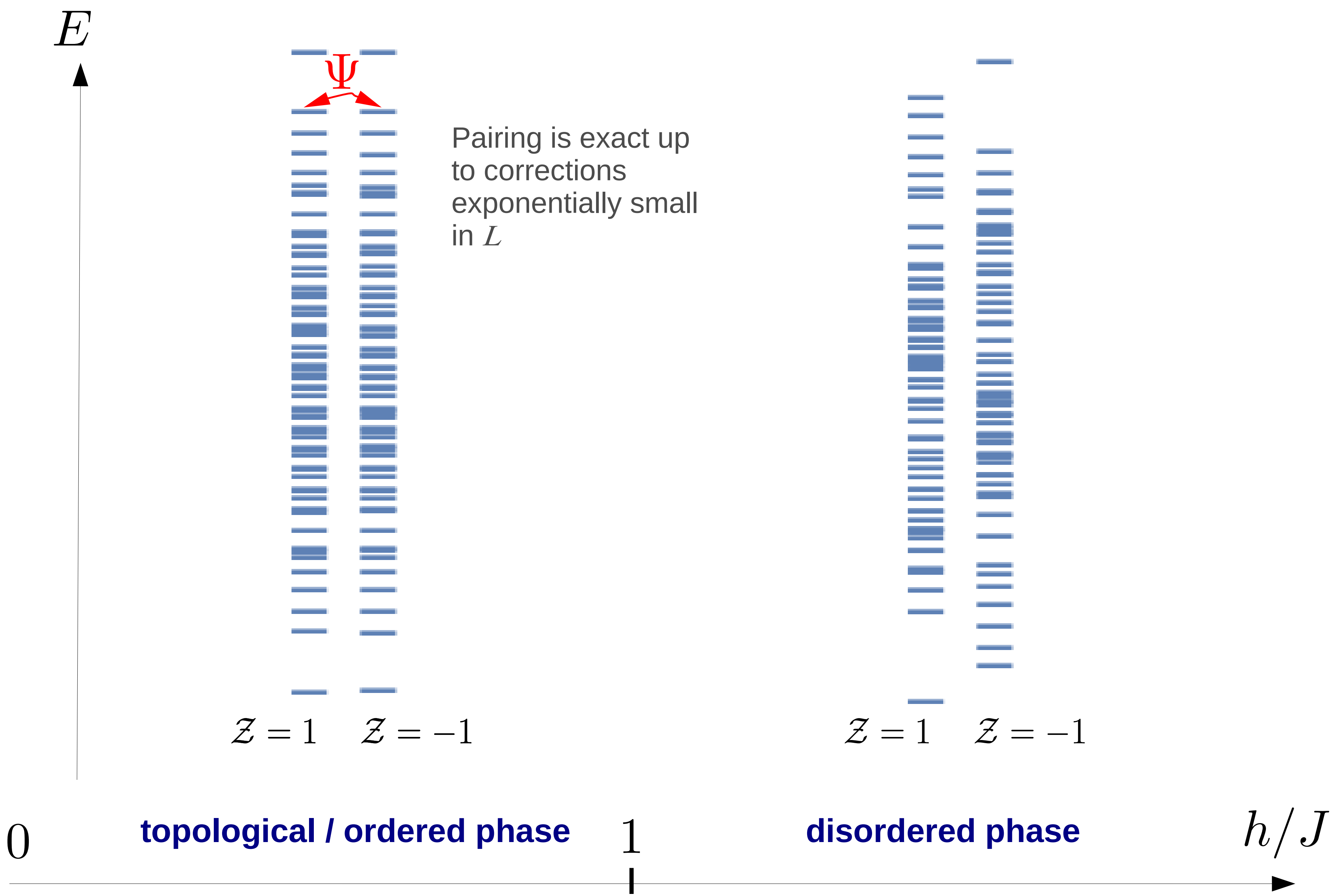
Properties of Ψ :

- **Commutes with H:** $[\Psi,H]=0$
- **Anticommutes with discrete symmetry :** $\{\Psi,Z\}=0$
- **Normalizable:** $\Psi^2 \propto 1$ (throughout the ordered phase)

0 **topological / ordered phase** 1 **disordered phase** h/J

Fendley '12

Consequences for the spectrum



Consequences for the physics

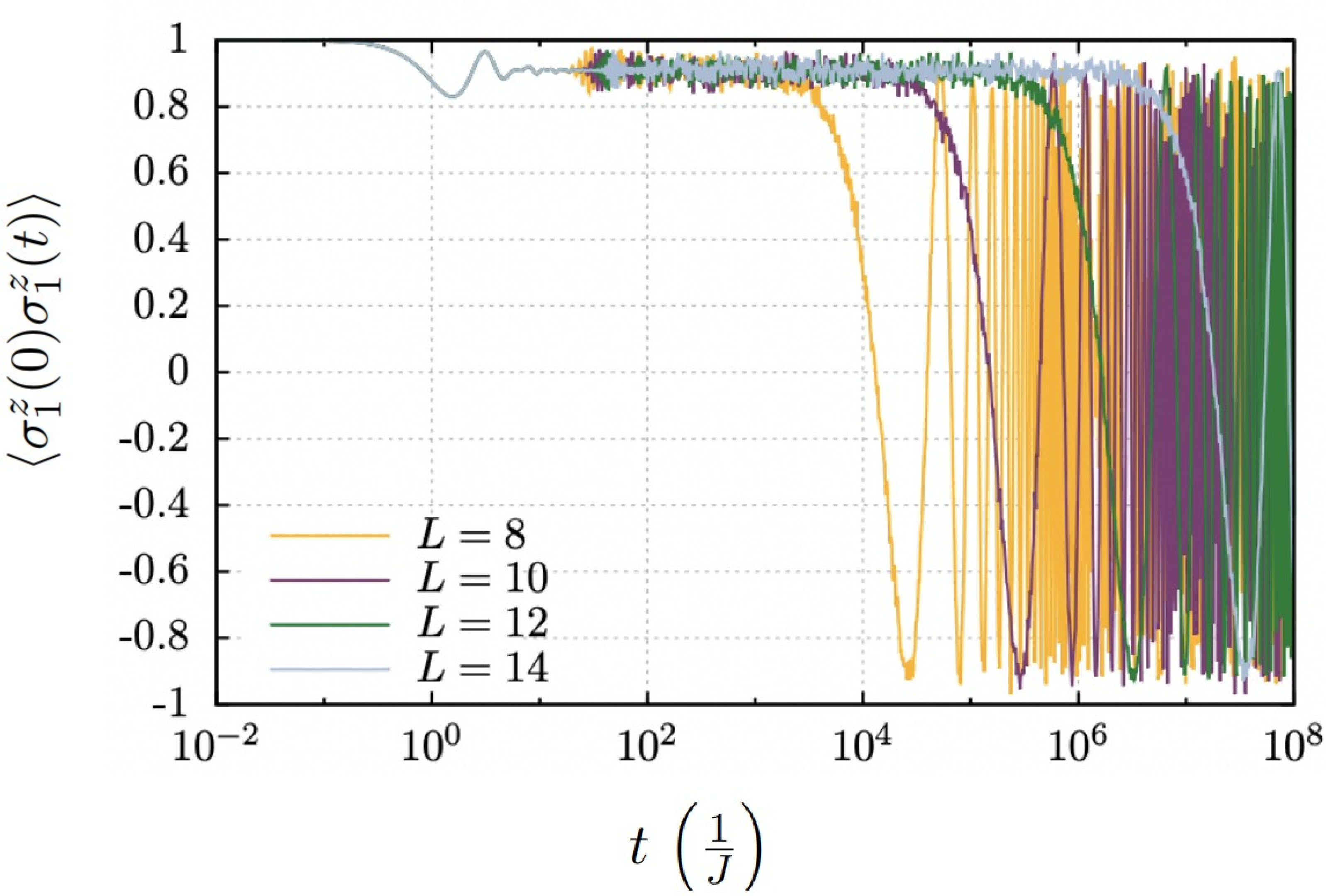


Figure from :

Kemp, Yao, Laumann & Fendley '17

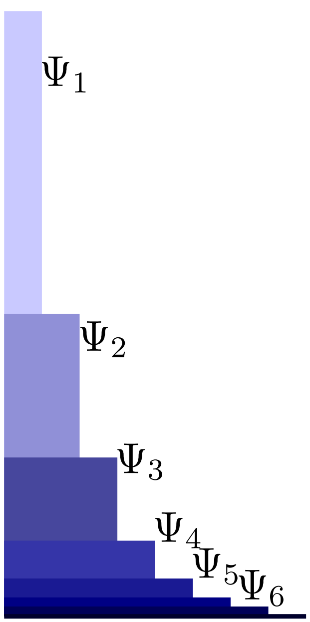
Very long coherence time : information is protected at the boundary !

SZM in XYZ

All this was for a model of noninteracting fermions.
Extension to an interacting model : **XYZ chain in its gapped phase** $(|J_z/ >|J_x| \,,\, |J_y|)$

$$H_{\text{free}} = \sum_{j=1}^{L-1} h_{j,j+1} \qquad h_{j,j+1} = J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z$$

Exact SZM constructed by Fendley (2015), from brute force calculation :



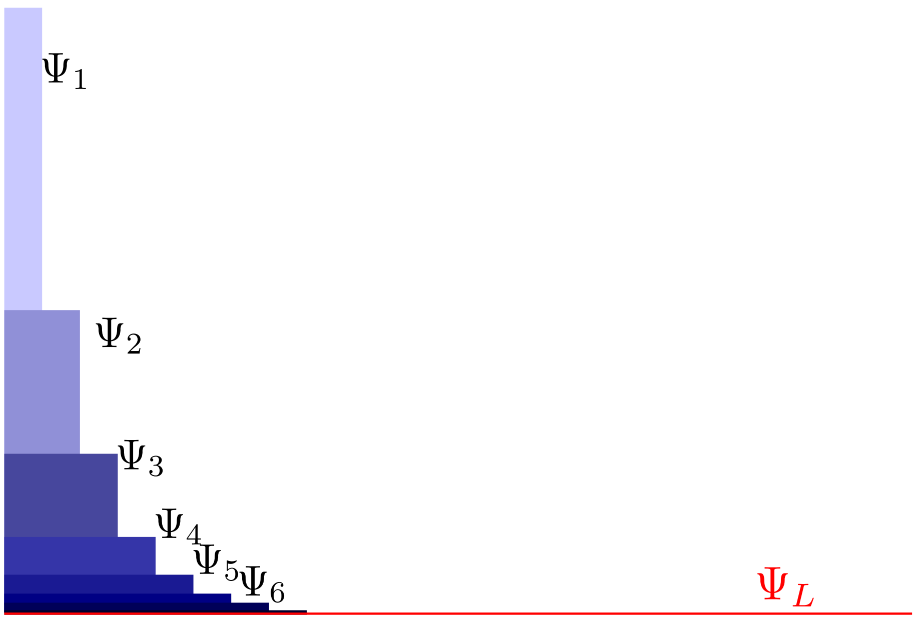
$$\Psi = \sum_{r \geq 1} \Psi_r$$

$$\Psi_r \equiv \sum_{0<j_1<\cdots<j_{2r}<r} \left(\frac{J_z^2}{J_x J_y}\right)^{-(k-1)} \sigma_r^z \prod_{k=1}^{r} \psi\big(j_{2k-1}, j_{2k}\big)$$

$$\psi(j,j') = \left(\frac{J_z}{J_y}\right)^{j'-j} \big(1 - \tfrac{J_z^2}{J_x^2}\big) \sigma_j^x \sigma_{j'}^x + \left(\frac{J_z}{J_x}\right)^{j'-j} \big(1 - \tfrac{J_z^2}{J_x^2}\big) \sigma_j^y \sigma_{j'}^y$$

ExactSZM in XYZ

SZM can be turned into an exact conserved quantity by adding boundary field at the far end



$$H = \sum_{j=1}^{L-1} h_{j,j+1} + \mathfrak{h}\sigma_L^z$$

$$\Psi = \sum_{r=1}^{L-1} \Psi_r + \frac{1}{\mathfrak{h}} \Psi_L$$

$$[H, \Psi] = 0$$

H integrable, but Z_2 symmetry is broken $\Rightarrow$ no pairing of energies Essler Fendley EV

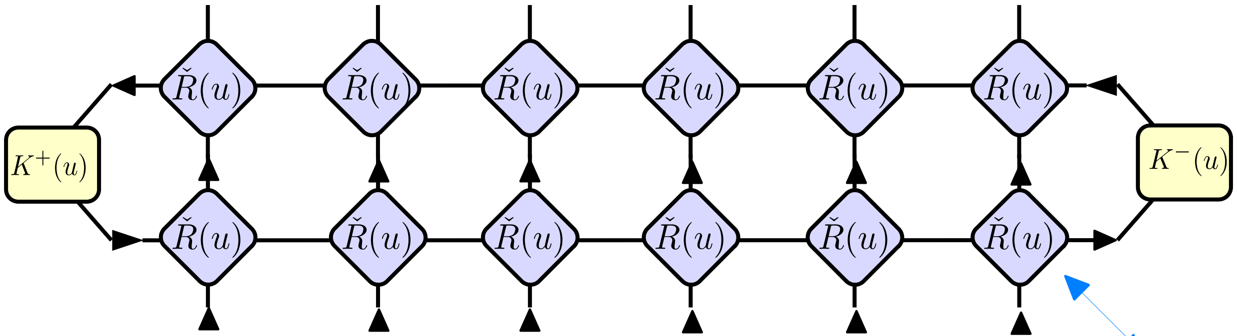
==> Relation with integrability ??

1. Boundary strong zero modes

2. Exact strong zero modes from integrability

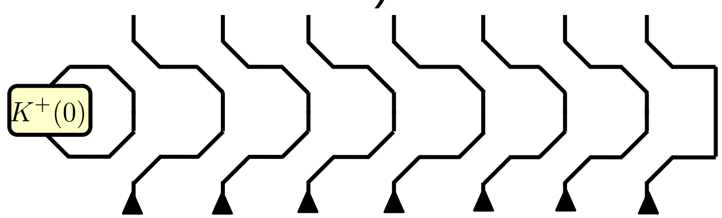
3. XYZ revisited

The usual integrability toolbox

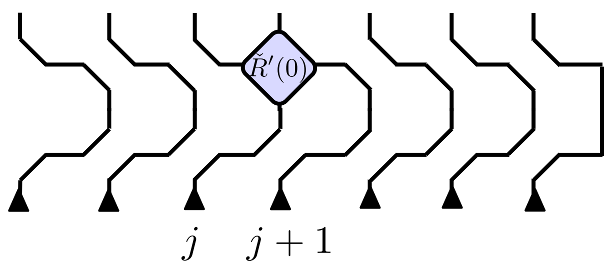
$T(u) =$ 

XYZ ↔ eight-vertex model
↔ elliptic weights

- $[T(u), T(v)] = 0$
- Expanding $T(u)$ around $u=0$ yields Hamiltonian and higher «local» conserved charges (homogeneous sum of local densities)

$T(0) =$ 

$\propto$ identity

$T'(0) \propto \sum_{j=1}^{L-1}$ 

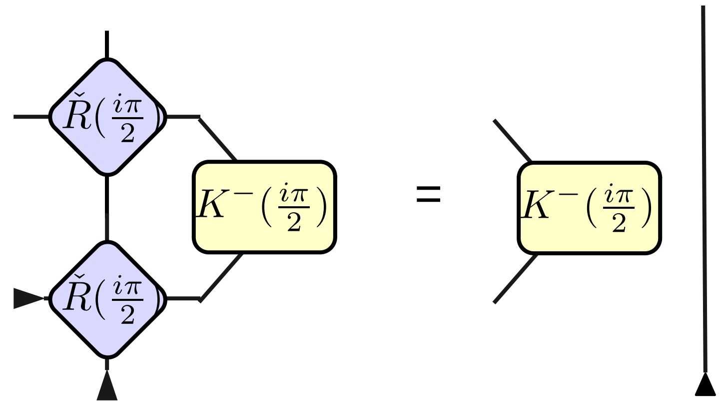
+ boundary terms

What we are after here is a different kind of object, exponentially localized at the boundary

EZSM from transfer matrix

EV Yeh Mitra Piroli ‘24

Key idea : expand not around $u=0$, but around $u=i\pi/2$



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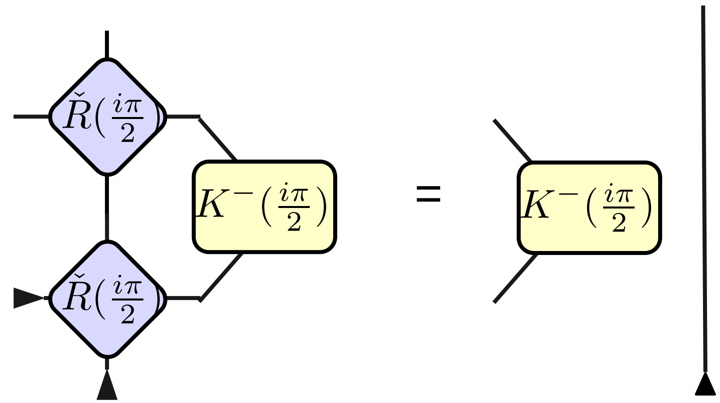
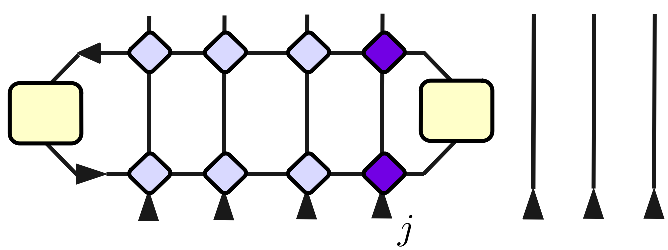


$\mathbf{T'}(i\pi/2) = \sum_{j=1}^{L}$

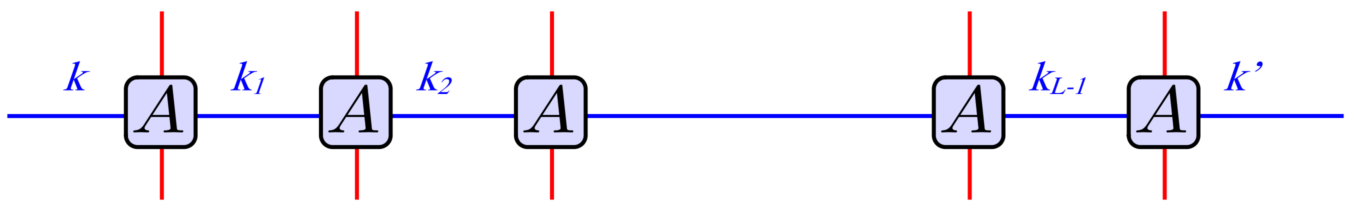


$= \sum_{j=1}^{L}$ term acting nontrivially on first j sites, with amplitude $\exp(-\ldots j)$

- $\boldsymbol{\Psi \propto \mathrm{T}'(i\pi/2)}$
- Proof that $\Psi^2 \propto 1$ becomes straightforward in MPO langage (brute force calculation in Fendley’s original paper)
- Conditions above can be readily adapted to other models or other geometries : higher spin XXZ Essler Fendley EV ; Floquet circuit EV Yeh Mitra Piroli ‘24

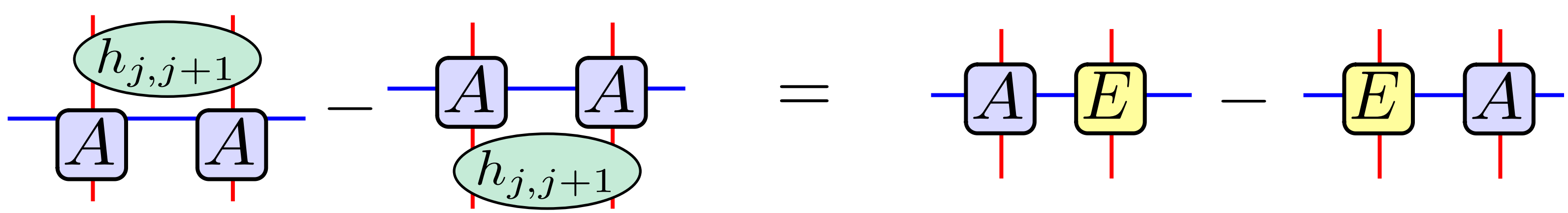
Independently, Fendley and Verstraete had derived a 4-dimensional MPO commuting with the XYZ Hamiltonian starting from the SZM [2018, unpublished]

$$\mathcal{M}_{k,k'} \equiv \sum_{\substack{\{a_j\},\{k_j\} \\ \in\{0,x,y,z\} \quad \in\{0,x,y,z\}}} A^{a_1}_{k k_1} A^{a_2}_{k_1 k_2} A^{a_3}_{k_2 k_3} \ldots A^{a_L}_{k_{L-1} k'} \, \sigma_1^{a_1} \sigma_2^{a_2} \ldots \sigma_L^{a_L}$$



Matrices A required to obey an equation of the form

$$[h_{j,j+1}, A \otimes A] = A \otimes E - E \otimes A$$



from which commutation with H follows telescopically.

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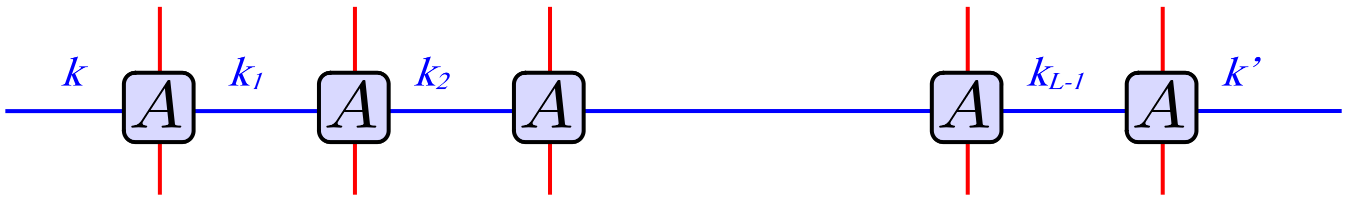


$$A^0_{00} = 1 \,, \qquad A^0_{kk} = vJ_k \,, \qquad A^k_{0k} A^k_{k0} = vJ_k - v^2 J_l J_m$$

$$\{k,l,m\} = \{x,y,z\}$$

« spectral parameter »

- MPOs commutes with Hamiltonian for any v
- MPOs with different v 's commute, as can be shown from the existence of a 16x16 R matrix
- Works for periodic case too

$$A^0_{00} = 1 \,, \qquad A^0_{kk} = vJ_k \,, \qquad A^k_{0k}A^k_{k0} = vJ_k - v^2 J_l J_m$$

$$\{k,l,m\} = \{x,y,z\}$$

It should come as no surprise that this 4-dimensional MPO coincides with the traditional open eight-vertex TM (or, in the periodic version, a product of two 8v TMs).

However, it is much nicer !!

- Very few non-zero coefficients
- No elliptic functions
- In the MPO, σ^k always come in pairs (with nothing in between)

Note : Katsura '14 : analogous MPO for XXX

1. Boundary strong zero modes

2. Exact strong zero modes from integrability

3. XYZ revisited

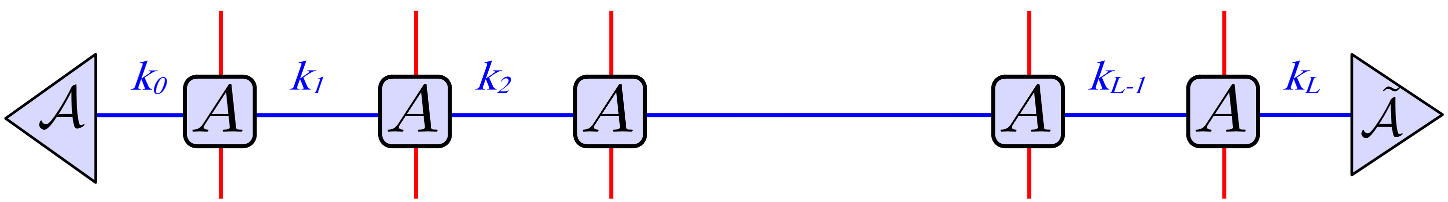
1) Integrable boundaries and impurities
Fendley Ghermann EV Verstraete 2025

2) Higher spin models
EV Fendley, in preparation

Boundaries

$$H = H_{\text{free}} + \sum_{b\in\{x,y,z\}} \left(\mathfrak{h}_b \sigma_1^b + \tilde{\mathfrak{h}}_b \sigma_L^b \right)$$

$$\mathcal{M} = \sum_{\{a_j\},\{k_j\}} \mathcal{A}_{k_0} A^{a_1}_{k_0 k_1} A^{a_2}_{k_1 k_2} A^{a_3}_{k_2 k_3} \dots A^{a_L}_{k_{L-1} k_L} \tilde{\mathcal{A}}_{k_L}\, \sigma_1^{a_1} \sigma_2^{a_2} \dots \sigma_L^{a_L}$$

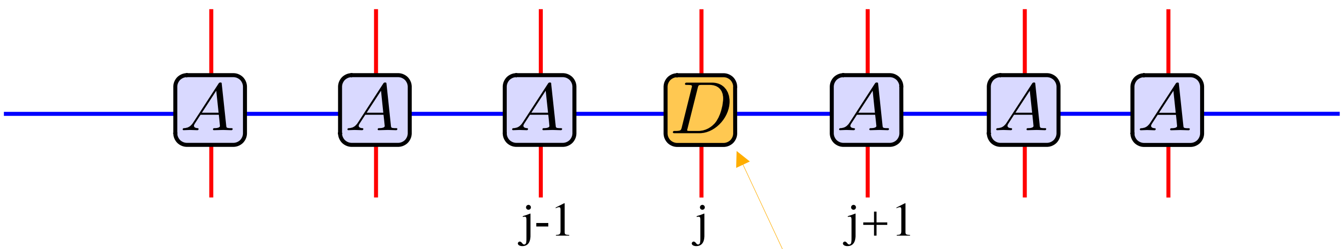


Requiring commutation with Hamiltonian fixes :

$$\frac{\mathcal{A}_b}{\mathcal{A}_0} = v \frac{\mathfrak{h}_b}{A^b_{b0}} \qquad\qquad \frac{\tilde{\mathcal{A}}_b}{\tilde{\mathcal{A}}_0} = v \frac{\tilde{\mathfrak{h}}_b}{A^b_{0b}}$$

→ Simple proof that XYZ is integrable for arbitrary boundary magnetic fields

... and impurities



unknown

$$H = \sum_{i<j-1} h_{i,i+1} + \boxed{h_{j-1j,j+1}} + \sum_{i>j} h_{i,i+1}$$

Commutation with the Hamiltonian requires:

$$[h_{j-1,j,j+1}, A \otimes D \otimes A] = A \otimes D \otimes E - E \otimes D \otimes A$$

- This yields a one-parameter family of impurity Hamiltonians.
- The same could be achieved, in principle, in the traditional approach by allowing for inhomogeneous spectral parameters. However much more cumbersome.
- Placing the impurity near the edge recovers a gapped lattice analog of the Kondo problem

Higher spin models

In XXZ, higher-spin models can be obtained by replacing Pauli matrices on physical sites by generators of $U_q(sl_2)$.

For XYZ, similarly, the required algebra was found by Sklyanin (1982) :

$$[S^0, S^k] = i\mathcal{J}_{lm}\{S^l, S^m\} \qquad \mathcal{J}_{lm} = \frac{J_m - J_l}{J_m + J_l}$$

$$[S^k, S^l] = i\{S^0, S^m\}$$

Quite remarkable how little is known about Sklyanin algebra, as opposed to $U_q(sl_2)$.

For instance, explicit matrix representations are written in the litterature for dim 2 and 3 only !!! (as far as we searched)

- in d=2 : Pauli matrices
- in d=3 :

$$S_0 = \begin{pmatrix} \mathcal{J}_z & 0 & \mathcal{J}_x - \mathcal{J}_y \\ 0 & \mathcal{J}_x + \mathcal{J}_y - \mathcal{J}_z & 0 \\ \mathcal{J}_x - \mathcal{J}_y & 0 & \mathcal{J}_z \end{pmatrix}, \quad S_x = \sqrt{2\mathcal{J}_y\mathcal{J}_z}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathcal{J}_k = J_k(J_l + J_m)$$

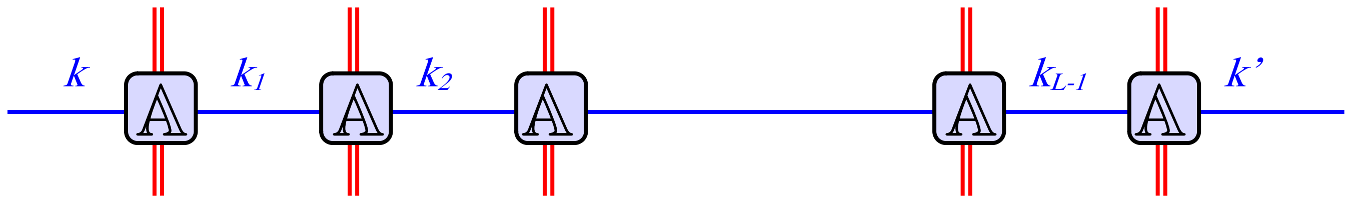
$$S_y = \sqrt{2\mathcal{J}_z\mathcal{J}_x}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad S_z = 2\sqrt{\mathcal{J}_x\mathcal{J}_y}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

As we will see our MPO makes the story quite nice, and worth revisiting

The higher spin MPO

EV Fendley, in preparation

$$\mathcal{M}_{kk'}(v) =$$



$$\mathbb{A}_{00} = K_0 + vB$$

$$\mathbb{A}_{kk} = v\beta_k + (J_k - vJ_lJ_m)B_k\,, \qquad \beta_k = J_kK_0 + \alpha J_lJ_m$$

$$\mathbb{A}_{0k} = 2\alpha_k\{S_0, S_k\}\,, \qquad \mathbb{A}_{k0} = 2\bar{\alpha}_k\{S_0, S_k\}\,, \qquad \alpha_k\bar{\alpha}_k = v(J_k - J_lJ_mv)$$

$$\mathbb{A}_{kl} = \frac{2\alpha_{kl}}{J_k + J_l}\{S_k, S_l\}\,, \qquad \alpha_{kl}\alpha_{lk} = (J_k - J_lJ_mv)(J_l - J_kJ_mv)$$

$$B_k = \frac{(S_k)^2 + (S_0)^2 - (S_l)^2 - (S_m)^2}{J_k}\,, \qquad B = \sum_{k=x,z,y}(J_k)^2 B_k$$

- One-parameter family of mutually commuting MPOs, as for XYZ (commutation follows from Sklyanin algebra, + same R matrix as before)

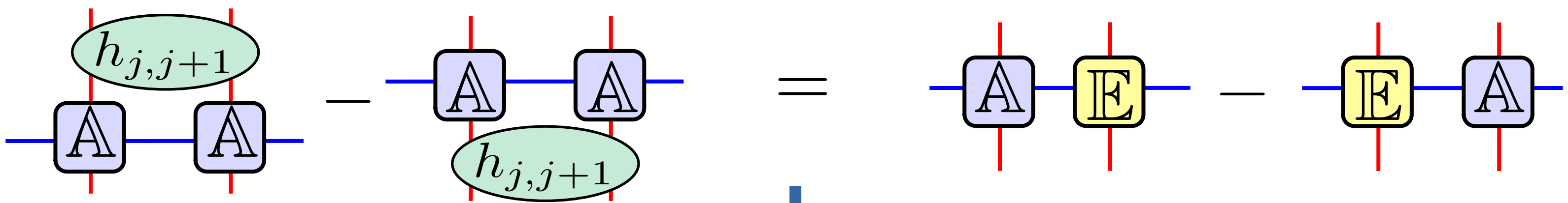
- Also expected to commute with some higher-spin Hamiltonian, which for $J_x = J_y$ reduces to the known «higher XXZ» Hamiltonians.

  Those have a known explicit form in terms of $U_q(sl_2)$ generators, however already quite complicated
  => can we find a simple form for the Hamiltonian ?

The Hamiltonian

$$H = \sum_j h_{j,j+1}$$

$$[h_{j,j+1}, \mathbb{A} \otimes \mathbb{A}] = \mathbb{A} \otimes \mathbb{E} - \mathbb{E} \otimes \mathbb{A}$$



$$[h, \sum_{\alpha=\{0,x,y,z\}} S^\alpha \otimes S^\alpha] = 0$$

$$[h, S^0 \otimes S^k + S^k \otimes S^0] = i(J_l - J_m)(S^l \otimes S^m + S^m \otimes S^l)$$

$$[h, i(S^l \otimes S^m - S^m \otimes S^l)] = (J_l + J_m)(S^0 \otimes S^k - S^k \otimes S^0)$$

- These characterize h uniquely (up to identity shift) in a given rep
- Not yet an explicit formula in terms of the S^a, but very nice and quite as practical (we hope to make further progress !)

More ongoing work :

- **Explicit matrix representations of Sklyanin generators**

  In progress. Interesting even/odd effect

- **Algebra of the MPOs**

  Fixing the left boundary index and letting the right index vary, the matrices $\mathrm{M}_{k}(v)=\mathrm{M}_{\cdot k}(v)$ obey some rather nice-looking algebra

$$[M_\alpha(u), M_\alpha(v)] = 0$$
$$(J_k - J_l J_m v)(J_l - J_k J_m u)[M_k(u), M_l(v)] = (J_k - J_l J_m u)(J_l - J_k J_m v)[M_k(v), M_l(u)]$$
$$u(J_k - J_l J_m v)[M_k(u), M_0(v)] = v(J_k - J_l J_m u)[M_k(v), M_0(u)]$$

$$(v-u) J_l J_m [M_k(u), M_0(v)] + iv(J_k - u J_l J_m)\epsilon^{klm}\{M_l(u), M_m(v)\} = 0$$

$$i(\{M_m(u), M_0(v)\} - \{M_m(v), M_0(u)\}) = \frac{J_k J_l (u-v)}{(J_k - v J_l J_m)(J_l - u J_l J_m)}[M_l(v), M_k(u)]$$

  This should be some elliptic deformation of the Onsager algebra (as the XXZ case is known to generate q-Onsager algebra Baseilhac Koizumi 2005)

Summary

At best, the 4-dimensional (=2-row) MPO is fundamentally simpler than the usual transfer matrix

At worst, just an opportunity to revisit XYZ and learn new things along the way :

- simple proof of integrability, including boundary fields and impurities
- Revisiting the very-under-studied Sklyanin algebra, simple characterization of the higher-spin Hamiltonian
- Algebra of the MPOs – construction of the eigenstates ? (Bethe ansatz without U(1))

Thank you for your attention !