UV Complete Theories from massless S-matrix Bootstrap

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Challenges in Integrability
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Integrability: StatMech vs QFT

	StatMech	QFT
defined by	Hamiltonian	Lagrangian
dof	spins	fields
integrability	R-matrix YBE	S-matrix YBE, Bootstrap
true dof	nonrel. quasi-particles	rel. particles and boundstates
eigenstate	Bethe ansatz Eq	Bethe-Yang Eq
TBA	thermodynamics	space$\leftrightarrow$ time imaginary time

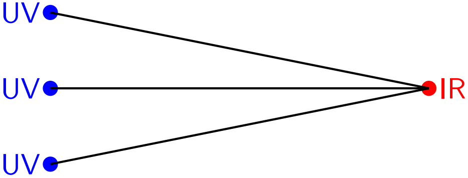
Introduction

- (Challenge) Find **all** QFTs which interpolate two RG fixed points

- IR fixed points (Wilson-Fisher) are important in CMP (ex) Kondo effects

- Dream of HEP is to find UV complete theories (ex) asymptotic safety

- (Modest Goal) Find all **2d integrable QFTs** which interpolate **a given IR CFT** with UV CFTs



Outline

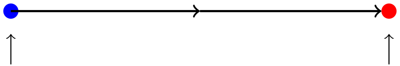
1. RG flows from IR to UV [LeClair, CA (2022)]
2. Massless S-matrix Bootstraps [CA (2025)]
3. TBA / Y-systems
4. Summary and Outlook

RG flows from IR to UV

[LeClair & CA, JHEP (2022)]

Conventional RG

- As scale changes from UV to IR, RG flow is generated and new IR fixed points may appear

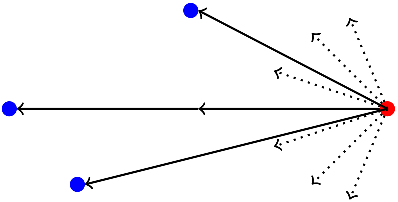


out along a **chosen** rel direction; in along a **resulting** irrel direction

- Consistent with the Zamolodchikov's C-theorem
- This type of RG flow is rare since the UV CFT has a limited set of relevant fields

RG from IR to UV

- Add some irrelevant fields to a CFT and increase (Energy) scale
- Need integrability to keep track on the RG flow (fine-tuning)
- New UV CFTs may be reached along some irrelevant directions



Integrable irrelevant deformations

- $T\overline{T}$: Recently active developments [Tateo,Zamolodchikov,...]
- Energy-momentum tensor T_2
- All higher conserved charges = $\{ \, [T\overline{T}]_s \, \}$

$$[T\overline{T}]_s = T_{s+1}\overline{T}_{s+1} - \Theta_{s-1}\overline{\Theta}_{s-1}, \quad \partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1}$$

- Preserve quantum integrability
- Exact results possible for even non-integrable theories

$T\overline{T}$-integrability [Smirnov-Zamolodchikov (2017)]

- Define an IQFT, with a known S-matrix $S_0(\theta)$, by

$$\text{IQFT} = \text{CFT} + \lambda\Phi_{rel}$$

- Deformations of this IQFT by $T\overline{T}$ fields preserve quantum integrability

$$\text{New IQFTs} = \text{IQFT} + \sum_{s=1}^{\infty} \alpha_s [T\overline{T}]_s$$

- Particle spectrum remain identical

- The S-matrix gets a CDD (scalar) correction

$$\mathrm{S}(\theta) = \mathrm{S}_{cdd}(\theta)\,\mathrm{S}_0(\theta), \qquad \mathrm{S}_{cdd}(\theta) = \prod_{s=1}^{\infty} e^{2i\alpha_s M^{2s}\sinh(s\theta)}$$

Massless limit

- Relativistic dispersion relation is given by the rapidity with mass $M \propto |\lambda|^{1/(2-\Delta_\Phi)}$

$$E(\theta) = M\cosh\theta, \qquad P(\theta) = M\sinh\theta$$

- Take $\lambda \to 0^-$ or $M \to 0$ limit and rescale the rapidity

$$M \to 0 \ \& \ \theta = \pm\Lambda + \hat{\theta} \quad \text{with finite} \quad Me^{\Lambda} = \mu \text{ as } \Lambda \to \infty$$

- R and L particles appear depending on $\pm$

$$(R) : E = P = \frac{\mu}{2}e^{\hat{\theta}}, \qquad (L) : E = -P = \frac{\mu}{2}e^{-\hat{\theta}}$$

Massless S-matrices

- From massive S-matrix $S_{cdd}(\theta)\, S_0(\theta)$ with $\theta = \theta_1 - \theta_2$
- RR, LL sectors with $\theta = \theta_1 - \theta_2 = (\hat{\theta}_1 \pm \Lambda) - (\hat{\theta}_2 \pm \Lambda) = \hat{\theta}$

$$S_{cdd}(\theta) = \prod_{s=1}^{\infty} e^{2i\alpha_s M^{2s} \sinh(s\theta)} \to 1 \quad \text{as} \quad M \to 0$$

- RL, LR sectors: S_0 becomes a phase as $\theta = \hat{\theta} \pm 2\Lambda$ and

$$\prod_{s=1}^{\infty} e^{2i\alpha_s M^{2s} \sinh(s\theta)} \to \prod_{s=1}^{\infty} e^{\pm i\alpha_s \mu^{2s} e^{\pm s\hat{\theta}}} \equiv \mathsf{S}_{cdd}(\pm\hat{\theta})^{\pm 1}$$

- Massless S-matrices with doubled spectrum (R and L)

$$\boxed{\mathsf{S}^{RR}(\theta) = \mathsf{S}^{LL}(\theta) = \mathsf{S}_0,\ \mathsf{S}^{RL}(\theta) = \mathsf{S}^{LR}(-\theta)^{-1} = \mathsf{S}_{cdd}(\theta)}$$

- Finding $S^{RL} = \mathsf{S}_{cdd}$ is the main challenge

Choose IR CFT

- Notation for CFTs in terms of coset CFT

$$[G]_k \equiv \frac{G_1 \otimes G_k}{G_{k+1}}, \qquad G_k = \text{WZW w. level k}$$

- For $G = A_n = su(n+1)$: $[A_n]_k = W_{n+1}$-minimal CFT series
- $[A_1]_k =$ minimal series: Ising, TIM, 3-state Potts,...
- This CFT deformed by a least relevant field is integrable

$$\text{IQFT} = [\text{G}]_\text{k} + \lambda \, \Phi_{\text{least rel}}$$

- (ex) $[E_8]_1$: Ising model with the spin field and E_8 scattering theory [Zamolodchikov]
- Will Consider $[A_n]_1$ for $n = 1, 2, \cdots$ for simplicity

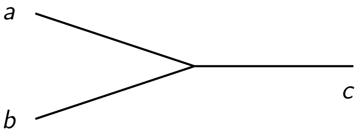
Massless S-matrix Bootstraps

[CA, JHEP (2025), 2509.20740]

Massive $[A_n]_1$ theory

- Every particle for IQFT is a boundstate of others

$$|A_c(\theta)\rangle = |A_a(\theta + i\overline{u}_{a\bar{c}}^b) \cdot A_b(\theta - i\overline{u}_{b\bar{c}}^a)\rangle, \quad a, b, c = 1, 2, \cdots, n$$



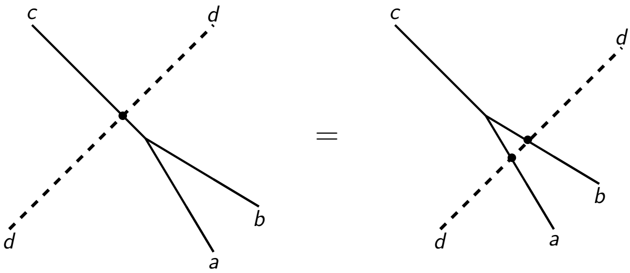
$$\overline{u}_{a\bar{c}}^b = \begin{cases} \frac{b\pi}{n+1}, & a + b = c \\ \frac{b\pi}{n+1}, & a + b = n + 1 + c \end{cases}, \quad \text{with } \bar{\mathrm{c}} \equiv \mathrm{n} + 1 - \mathrm{c}$$

- Diagonal S-matrix is defined by

$$A_a(\theta_1)A_b(\theta_2) = S_{ab}(\theta_1 - \theta_2)A_b(\theta_2)A_a(\theta_1)$$

Massive Bootstrap Eq

- graphically



- Bootstrap equations

$$\boxed{S_{dc}(\theta) = S_{da}(\theta - i\overline{u}^{b}_{a\overline{c}})\, S_{db}(\theta + i\overline{u}^{a}_{b\overline{c}})}$$

Massive solution

- Unique solution have been found [Körberle-Swieca (1979)]

$$S_{ab}(\theta) = (a+b)_-(a+b-2)_-^2 \cdots (|a-b|-2)_-^2(|a-b|)_-$$

where

$$\boxed{(k) = \frac{\sinh\left[\frac{1}{2}\left(\theta - \frac{ik\pi}{n+1}\right)\right]}{\sinh\left[\frac{1}{2}\left(\theta + \frac{ik\pi}{n+1}\right)\right]}}, \qquad (k)_- = (-k) = \frac{1}{(k)}$$

- For $k > 0$, $(k)_-$ has a pole at $\theta = ik\pi/(n+1)$ in physical strip

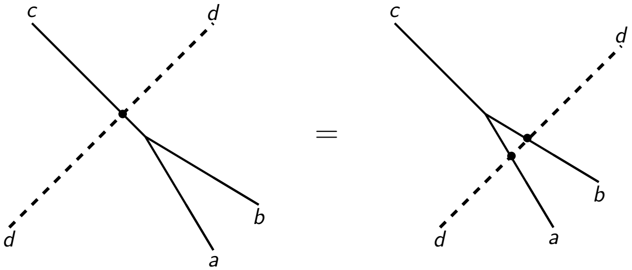
Massless $[A_n]_1$ theory

- Taking the massless limit $\theta \to \theta \pm \Lambda$,

$$|A_c^{R,L}(\theta)\rangle = |A_a^{R,L}(\theta + i\overline{u}_{a\bar{c}}^{b}) \cdot A_b^{R,L}(\theta - i\overline{u}_{b\bar{c}}^{a})\rangle, \quad a, b, c = 1, 2, \cdots, n$$

- Massless Bootstrap equations $(a, b, c, d = 1, 2, \cdots, n)$

$$\boxed{S_{dc}^{RL}(\theta) = S_{da}^{RL}(\theta - i\overline{u}_{a\bar{c}}^{b})\, S_{db}^{RL}(\theta + i\overline{u}_{b\bar{c}}^{a})}$$



- No poles in S^{RL}-matrices $\to$ more solutions possible

Defining relations

- Some of $\mathcal{O}(n^4)$ bootstrap equations can define all $S_{ab}(\theta)$ (omit RL)

$$S_{dc}(\theta) = S_{da}(\theta - i\overline{u}^{b}_{a\overline{c}})\, S_{db}(\theta + i\overline{u}^{a}_{b\overline{c}})$$

- Starting with $a = 1, b = 1 \to c = 2$ along with $d = 1$, we can obtain S_{12} from S_{11}. **All S_{ab} are decided by S_{11}.**

$$\begin{array}{ccccccccc}
S_{11} & \to & S_{12} & \to & S_{13} & \to & \cdots & \to & S_{1n} \\
 & & \downarrow & & & & & & \\
 & & S_{22} & \to & S_{23} & \to & \cdots & \to & S_{2n} \\
 & & & & \downarrow & & & & \\
 & & & & S_{33} & \to & \cdots & \to & S_{3n} \\
 & & & & & & \cdots & & \vdots \\
 & & & & & & & & S_{nn}
\end{array}$$

Constraints

- The remainig $\mathcal{O}(n^4 - n^2)$ bootstrap equations are constraints
- The crossing-unitarity relations also should be satisfied

$$S^{RL}_{ab}(\theta)S^{RL}_{\bar{a}b}(\theta + i\pi) = 1$$

- The remaining unknown amplitude is S^{RL}_{11}
- We find n different solutions of these constraints

$$S^{[m]}_{11} = e^{\frac{i\pi m_2}{n+1}}\,(m), \quad m = 1, 2, \cdots, n$$

- All S-matrices belonged to above $S^{[m]}_{11}$ are

$$\boxed{S^{[m]}_{ab} = e^{\frac{i\pi abm_2}{n+1}} \prod_{k=1}^{\min(a,m)} \left[\prod_{\ell=k}^{a+m-k} (a + b + m - 2\ell) \right]}$$

- Products of solutions $S^{[m_1]}_{ab} \cdot S^{[m_2]}_{ab} \cdots$ also possible

UV Complete theories

- We have restricted S^{RL} from infinite possibilities to only n choices (in this case the rank of A_n)

$$\mathrm{S}_{cdd}[\{\alpha_i\}] \longrightarrow S_{ab}^{[m]}, \quad m = 1, 2, \cdots, n$$

- These can be further reduced by a symmetry (related to the automorphism of A_n)

$$S_{ab}^{[1]} = S_{a\bar{b}}^{[n]}, \qquad S_{ab}^{[m]} = S_{a\bar{b}}^{[\overline{m}]}$$

- We find that UV complete theories are possible only for

$$S_{ab}^{[1]}, \quad S_{ab}^{[2]}, \quad S_{ab}^{[1]} S_{ab}^{[n]}$$

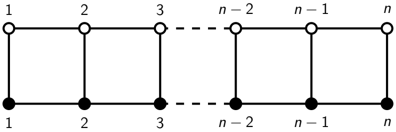
TBA / Y-systems

[CA, JHEP (2025), 2509.20740]

Minimal RG flows

- With $S^{RL} = S^{[1]}_{ab} = e^{\frac{i\pi ab}{n+1}}(a+b-1)(a+b-3)\cdots(|a-b|+1)$
- Y-system or Universal TBA

$$Y^R_a\left(\theta + \frac{i\pi}{n+1}\right)Y^R_a\left(\theta - \frac{i\pi}{n+1}\right) = \frac{\prod_{b=1}^{n}(1+Y^R_b(\theta))^{\mathbb{I}_{ab}}}{(1+1/Y^L_a(\theta))}, \quad (L \leftrightarrow R)$$



- This gives UV CFT $[A_n]_2$ connected to $[A_n]_1$ IR CFT

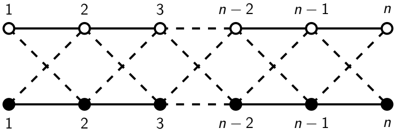
$$c_{UV} = c\left([A_n]_2\right), \qquad \Delta_\Phi = \frac{6}{n+4} = \Delta(\Phi_{\text{least rel}})$$

Diagonal RG flows

- With $S^{RL} = S^{[2]}_{ab} = 1/S^{RR}_{ab}$

- Y-system or Universal TBA

$$Y^R_a\left(\theta + \frac{i\pi}{n+1}\right) Y^R_a\left(\theta - \frac{i\pi}{n+1}\right) = \prod_{b=1}^{n} \left[\frac{1 + Y^R_b(\theta))}{1 + 1/Y^L_b(\theta)}\right]^{\mathbb{I}_{ab}}, \quad (L \leftrightarrow R)$$



- This gives UV CFT $su(n+1)_2/u(1)^n$ connected to $[A_n]_1$ IR CFT

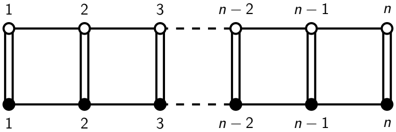
$$c_{UV} = \frac{n(n+1)}{n+3} = c\left(su(n+1)_2/u(1)^n\right), \qquad \Delta_\Phi = 2$$

Saturated RG flows

- With $S^{RL} = S_{ab}^{[1]} \cdot S_{ab}^{[n]} = S_{ab}^{[1]} \cdot S_{a\bar{b}}^{[1]}$

- Y-system or Universal TBA

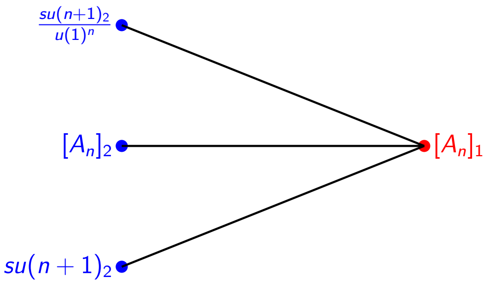
$$Y_a^R(\theta + \frac{i\pi}{n+1}) Y_a^R(\theta - \frac{i\pi}{n+1}) = \frac{\prod_{b=1}^{n} (1 + Y_b^R(\theta))^{\mathbb{I}_{ab}}}{(1 + 1/Y_a^L(\theta))(1 + 1/Y_{\bar{a}}^L(\theta))}$$



- This gives UV CFT $su(n+1)_2$ WZW connected to $[A_n]_1$ IR CFT

$$c_{UV} = c\left(su(n+1)_2\right), \qquad \Delta_\Phi = 2$$

Three RG flows



n	c_{IR}	c_{UV} (minimal)	c_{UV} (diagonal)	c_{UV} (saturated)
1	$1/2$	$7/10$	$1/2$	$3/2$
2	$4/5$	$6/5$	$6/5$	$16/5$
3	1	$11/7$	2	5
4	$8/7$	$13/7$	$20/7$	$48/7$

(Ex) Saturated RG flow for $n = 1$

Super-sinh-Gordon model [Kim, Rim, Al. Zamolodchikov, CA (2002)]

- Lagrangian of this SShG model is given by

$$\mathcal{L} = \mathrm{Kin.} - \frac{i}{2}\psi\overline{\psi}W''(\phi) + 2\pi[W'(\phi)]^2, \quad W(\phi) = -\mu \sinh\phi$$

- (cf) Conventional SShG model with $W(\phi) = \mu\cosh\phi$ where SUSY is unbroken
- Supersymmetry is spontaneous broken

$$V(\phi) \propto [W'(\phi)]^2 = \cosh^2\phi > 0$$

- Massless Goldstino fermion (both R and L) is only spectrum in the IR limit (Ising model)

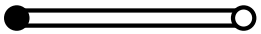
S-matrix and TBA

- The S-matrices between these chiral fermions are

$$S^{LL} = S^{RR} = -1, \qquad S^{RL} = -(1)^2 = -\tanh^2\left(\frac{\theta}{2} - \frac{i\pi}{4}\right)$$

- TBA also matches with the saturated RG flow of $n = 1$

$$Y^R(\theta + \frac{i\pi}{2})Y^R(\theta - \frac{i\pi}{2}) = (1 + 1/Y^L(\theta))^{-2}$$

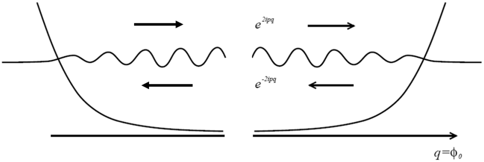


Momentum quantization condition

- Mini-superspace computation usning the SShG model Lagrangian

$$\delta(p) = \pi + 3p \ln \frac{R}{2\pi} = \sum_{n=\mathrm{odd}} \delta_n p^n, \quad \mathcal{R}(p) = e^{i\delta(p)}$$

where $\mathcal{R}(p)$ is the reflection amplitude of the super-Liouville theory



Reflection amplitude vs. the saturated TBA

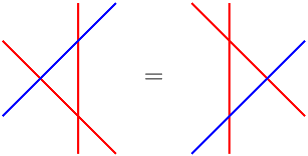
- Both match very well
- For comparison, we can reconstruct δ_n's from numerical TBA solutions using

$$c_{\text{eff}} = \frac{3}{2} - 24p^2 + \mathcal{O}(R) = c_{TBA}(R)$$

	δ_1	δ_3	δ_5
Reflection	1.42998	1.60274	0.829542
TBA	1.42998	1.60274	0.829567

Summary

- RG flows from IR CFT can be generated by integrable $T\overline{T}$ deformations
- Only a few UV complete theories are possible
- For diagonal scattering theories, we have used the massless S-matrix bootstrap methods to find complete solutions. The resulting RG flows are belonged to three categories; minimal, diagonal, and saturated. Except the minimal flows, others are mostly new.
- For nondiagonal S-matrix theories, S^{RL} can be found from YBEs [Bajnok-CA, JHEP (2024)]



$$S^{LL}S^{RL}S^{RL} = S^{RL}S^{RL}S^{LL}$$

Outlook

- Identifying IQFTs for diagonal and saturated flows
- Extending to other IR CFTs (D, E, . . .)
- There are conjectured RG flows based on TBAs, NLIEs, which can be justified by finding S-matrices.
- Non-invertible symmetries can be extended into IQFTs which show RG flows between two CFTs. [Tanaka-Nakayama (2024)] Can we find RG invariant quantities related to this in our three flows?
- Is there similar 2d integrable lattice model?
  (cf) [Pearce, Chim, CA (2003)] ABF model Regime IV

Thank you for attention!