Challenges in integrability（Wu Han 2025）

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Exact solution of a quantum integrable system associated with the $G2$ exceptional Lie algebra

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01 Quantum integrable model

Quantum integrable model

Quantum integrablility: A quantum stationary state system with degree N of freedom possesses N independent and mutually commuting mechanical quantities.

$$[L_i, L_j] = 0, (i, j = 1..., N)$$

$$R_{12}(k_1-k_2)R_{13}(k_1-k_3)R_{23}(k_2-k_3) = R_{23}(k_2-k_3)R_{13}(k_1-k_3)R_{12}(k_1-k_2)$$

$$T(u) = R_{01}(u)R_{02}\cdots R_{0N}(u) \quad t(u) = tr_0(T_0(u))$$

$$[t(u), t(v)] = 0$$

$$\mathcal{A}_{n-1}^{(1)},\ \mathcal{B}_n^{(1)},\ \mathcal{C}_n^{(1)},\ \mathcal{D}_n^{(1)},\ \mathcal{A}_{2n}^{(2)},\ \mathcal{A}_{2n-1}^{(2)},\ \text{and}\ \mathcal{D}_{n+1}^{(2)}$$

$$t(u) = \sum_{n=0}^{\infty} t^{(n)} u^n$$

$$[t^m, t^n] = 0 \quad [H, t^n] = 0 \quad H = \frac{1}{2}\sum_{j=1}^{N} \sigma_j \cdot \sigma_{j+1} = \frac{\partial \ln t(u)}{\partial u}\bigg|_{u=0,\{\theta_j=0\}} - \frac{1}{2}N,$$

Yang C N P.R.L, 1967, 19(23): 1312; Baxter R J Academic Press, 1982

Applications

Classical and quantum statistical mechanics,
Quantum field theory,
Quantum many-particle and condensed matter physics
Quantum optics
String theory and cosmology

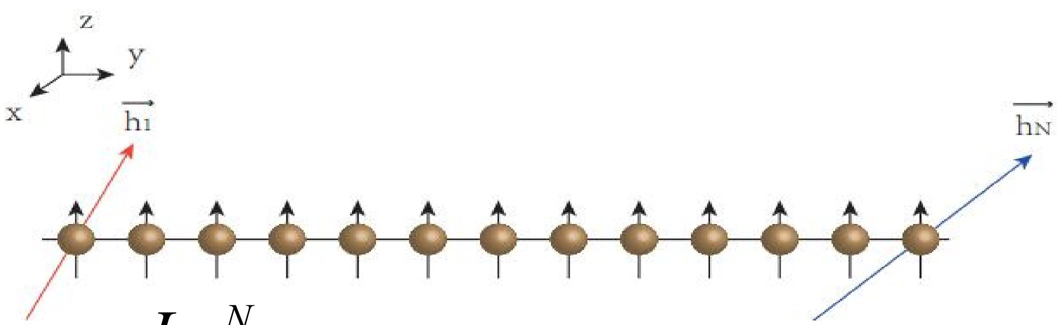
Some prominent examples of integrable models :

(1)various variants of the Heisenberg quantum spin chain whose physical realizations are probed by neutron scattering;
(2)the Hubbard model and its variants which inter alia have been discussed in connection with high-temperature superconductivity;
(3)the Kondo model which has recently seen a renaissance because of the development of tunable quantum dots;
(4) interacting Bose and Fermi gases which can now be produced in very pure and tunable form in optical lattices.

One-dimension spin chain

$$H = -\sum_{n=1}^{N} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cosh \eta \sigma_n^z \sigma_{n+1}^z),$$

$$\sigma_{N+1}^x = \sigma_1^x, \quad \sigma_{N+1}^y = -\sigma_1^y, \quad \sigma_{N+1}^z = -\sigma_1^z.$$



$$H = \frac{J}{2} \sum_{j=1}^{N} \sigma_j \bullet \sigma_{j+1} + h_1 \sigma_1^z + h_{Nz} \sigma_N^z + h_{Nx} \sigma_N^x$$

02

ODBA method

Exact solution

(1) Coordinate Bethe Ansatz,

Bethe H ,Zeitschrift fur Physik 1931,71(205):13

(2) Algebraic Bethe Ansatz ,

Faddeev L et al. Soviet Physics Doklady:1978,243:902-904,
V.A. Tarasov, Theor. Math. Phys. 56 (1988) 793
M.J. Martins, Nucl. Phys. B 450 (1995) 768.
S Belliard, N A Slavnov, B Vallet, Modified Algebraic Bethe Ansatz *SIGMA 14 (2018), 054*

(3) Analytical Bethe Ansatz,

Baxter R J, Annals of Physics 1972,281:187-222;
N. Yu. Reshetikhin,Theor. Math. Phys. 63 (1985) 555–569.
L. Mezincescu, R.I. Nepomechie, J. Phys. A 24 (1991) L17; Nucl. Phys. B 372 (1992) 597.

(4) Off - diagonal Bethe Ansatz,

Y. Wang, W. -L. Yang, J. Cao and K. Shi, Off-Diagonal Bethe Ansatz for Exactly Solvable Models, Springer Press (2015)

Off-diagonal Bethe-Ansatz(ODBA)

2013 year， Wang, Yang, Cao and Shi proposed OBDA :

An degree N polynomial requires N+1 conditions to be determined. If sufficient independent definite conditions can be found, then the polynomial can be completely determined. The central idea of the ODBA is to derive the functional *T* - *Q* relation based on *N* operator identities which determine the *N* unknowns of *Λ(u)* .

$$\Lambda(u) = \Lambda_0\{\prod^{N} f(u-u_j)\}$$

$$T_0(u) \;=\; R_{0,N}(u-\theta_N)\cdots R_{0,1}(u-\theta_1)$$

$$t(u) = tr_0 T_0(u) \quad H = J\frac{\partial lnt(u)}{\partial u}|_{u=0,\{\theta_j=0\}} - \frac{NJ}{2}$$

$$t(\theta_j)t(\theta_j-1) = a(\theta_j)d(\theta_j-1), \quad j=1,\cdots,N$$

$$a(u) = \prod_{j=1}^{N}(u-\theta_j+1), \quad d(u) = \prod_{j=1}^{N}(u-\theta_j)$$

$$a(\theta_j-1) = 0, \quad d(\theta_j) = 0$$

02

Off-diagonal Bethe-Ansatz(ODBA)

Traditional T-Q relation

$$\Lambda(u) = a(u)\frac{Q(u-1)}{Q(u)} + d(u)\frac{Q(u+1)}{Q(u)}$$

$$Q(u) = \prod_{j=1}^{N}(u-u_j) \quad t(u)|_{u\to\infty} = 2u^N \times id + \cdots$$

OBDA T-Q relation：

$$\Lambda(u) = e^{i\phi}a(u)\frac{Q(u-1)}{Q(u)} + e^{-i\phi}d(u)\frac{Q(u+1)}{Q(u)} + 2(1-cos\phi)\frac{a(u)d(u)}{Q(u)}$$

The inclusion of the third term is a significant breakthrough! It can not only work on models with broken U(1) symmetry, but also apply to models where U(1) symmetry remains unbroken

$$e^{i\phi}a(u_j)Q(u_j-1) + e^{-i\phi}d(u_j)Q(u_j+1) + 2(1-cos\phi)a(u_j)d(u_j) = 0$$

Achievements

(1) The XXZ model with anti-periodic boundary conditions has been solved；

Cao J, Yang W L, Shi K, Wang Y. Off-diagonal Bethe Ansatz and exact solution of a topological spin ring[J]. Physical Review Letters, 2013, 111: 137201-137205.

(2) Spin-1/2 XXX model with arbitrary boundary has been solved；

Cao J, Yang W L, Shi K, Wang Y. Off-diagonal Bethe Ansatz solution of the XXX spin chain with arbitrary boundary conditions[J]. Nuclear Physics B, 2013, 875(1): 152-165.

(3) The Su(n) spin chain model with arbitrary boundary conditions has been solved；

Cao J, Yang W L, Shi K, Wang Y. Nested off-diagonal Bethe Ansatz and exact solutions of the su(n) spin chain with generic integrable boundaries[J]. Journal of High Energy Physics, 2014, 04: 143-171.

(4) The Hubbard model with arbitrary boundary conditions has been solved；

Li Y Y, Cao J, Yang W L, Shi K, Wang Y. Exact solution of the one-dimensional Hubbard model with arbitrary boundary magnetic fields[J]. Nuclear Physics B, 2014, 879: 98-109.

(5) The XYZ spin chain model with odd-numbered lattice sites has been solved；

Cao J, Cui S, Yang W L, Shi K, Wang Y. Spin-1/2 XYZ model revisit: general solutions via off-diagonal Bethe Ansatz[J]. Nuclear Physics B, 2014, 886: 185-201.

Question : Is the ODBA method applicable to integrable models associate with B, C, and D Lie algebra?

Examples of ODBA

01 We have solved the B2 model. With the help of its vector representation and the spinor representation R matrix, we obtain seven different operator identities.

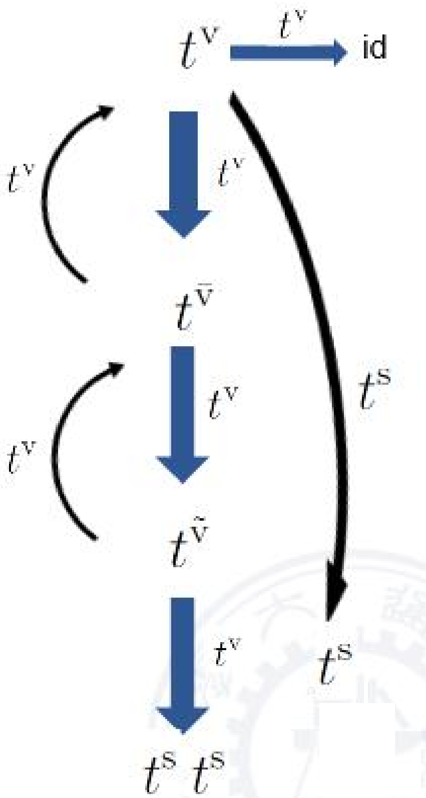
02 For Cn model, we need 2n-1 different operator identities to provide the operator closure relations.

03 To solve D3 model, besides the vector representation R matrix, we also need two nonequivalent spinor representation R matrix. The number of operator identities are four.

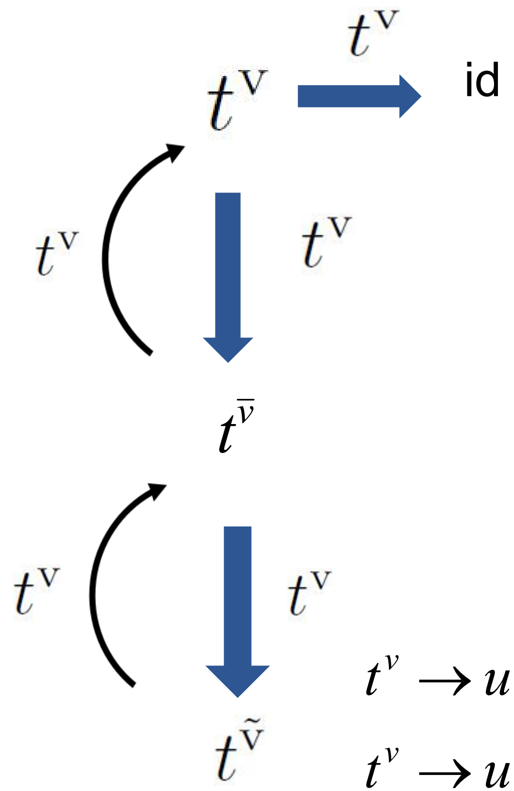
04 For D2 model, there are no enough operator identities due to its special degenerate point. But it can be decomposed into six vertex models.

02

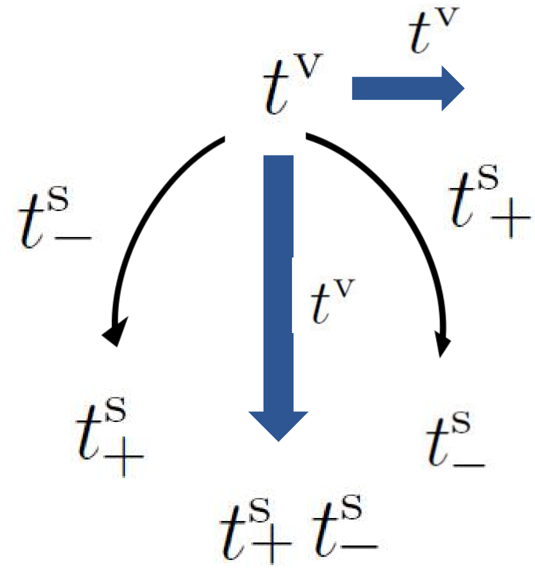
C2,C3,D3 Operator closure relation



B2



C3



D3

$$t^{v} \to u^{2N}, t^{\bar{v}} \to u^{2N}, t^{\tilde{v}} \to u^{2N}, t^{s} \to u^{N}, B_2$$

$$t^{v} \to u^{2N}, t^{\bar{v}} \to u^{2N}, t^{\tilde{v}} \to u^{N}, C_2$$

$$t^{v} \to u^{2N}, t^{s}{}_{+} \to u^{N}, t^{s}{}_{-} \to u^{N}, D_3$$

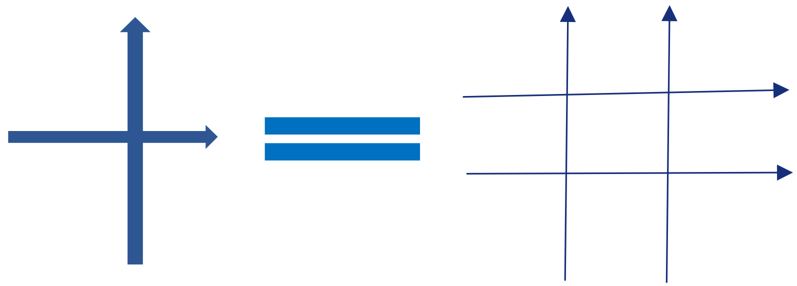
B2 NPB 946 (2019) 114719
C2 JHEP 05 (2019) 067; $A_3^{(2)}$ CMP 399 (2023) 651; Cn NPB 965 (2021) 115333
D3 JHEP 12 (2019) 051; $D_3^{(2)}$ JHEP 03 (2022) 175

$D_2^{(2)}$ Decomposition

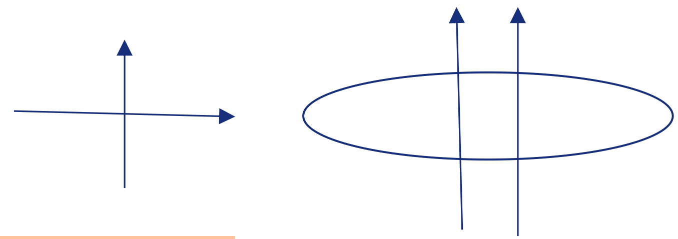
JHEP 04（2022）101

$$R^d_{12}(u) = 2^4 S_{1'2'} S_{3'4'} R_{1'4'}(u+i\pi) R_{1'3'}(u) R_{2'4'}(u) R_{2'3'}(u-i\pi) S^{-1}_{1'2'} S^{-1}_{3'4'}$$

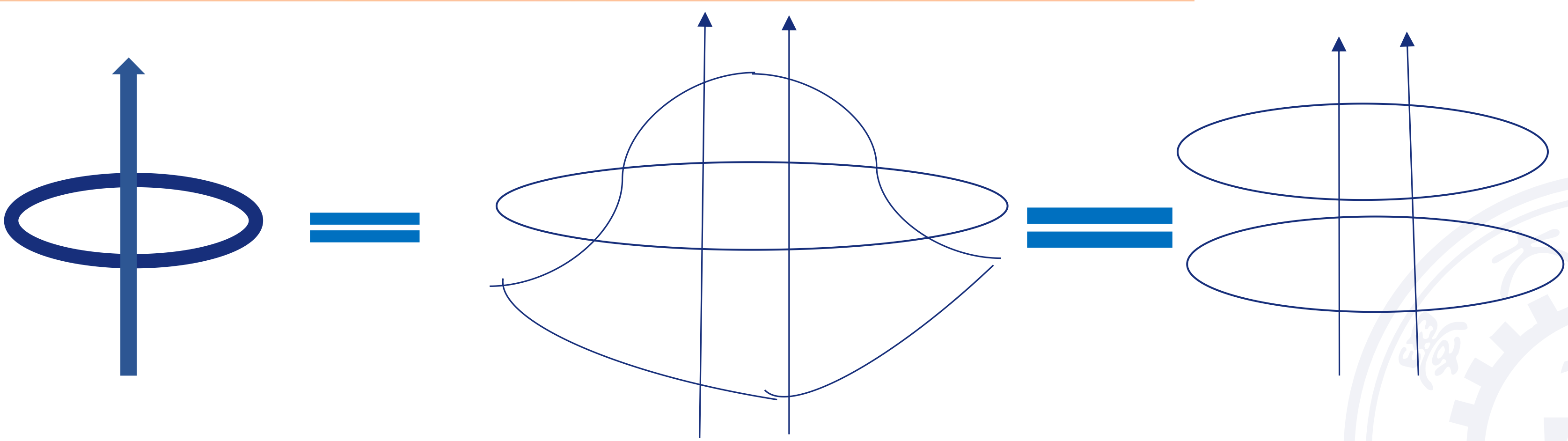
R.I. Nepomechie and A.L. Retore, *JHEP* **03** (2021) 089



$$t(u) = tr_{0'}\{\bar K_{0'}(u) T_{0'}(u) K_{0'}(u) \hat T_{0'}(u)\}$$



$$K_1^+(u) = [\rho_s(i\pi)]^{-\frac{1}{2}} S \tilde R_{2'1'}(i\pi) \tilde K^+_{2'}(u) \tilde M^{-1}_{2'} \tilde R_{1'2'}(-2u+4\eta-i\pi) \tilde M_{2'} \tilde K^+_{1'}(u) S^{-1},$$
$$K_1^-(u) = [\rho_s(i\pi)]^{-\frac{1}{2}} S \tilde K^-_{1'}(u+i\pi) \tilde R_{2'1'}(2u+i\pi) \tilde K^-_{2'}(u) \tilde R_{1'2'}(-i\pi) S^{-1},$$



$$t^d(u) = tr_0\{\bar K^d_0(u) T^d_0(u) K^d_0(u) \hat T^d_0(u)\}.$$

$$\bar t(u+i\pi)\bar t(u) = 2^{-8N}[\rho_s(2u+2i\pi-2\eta)]^{-1} \mathcal{S}^{-1} t^d(u) \mathcal{S},$$

03 Exact solution for G2

G2 model

M.J. Martins, NPB 989 (2023) 116131.
C.M. Yung, M.T. Batchelor, PLA 198 (1995) 395.

NPB 1010 (2025) 116777

$$\{H_1, H_2, E_1, \cdots, E_6, E_1^\dagger, \cdots, E_6^\dagger\}$$

$$E_3 = \frac{1}{\sqrt{2}}[E_1, E_2], \quad E_4 = \sqrt{\frac{3}{8}}[E_1, E_3], \quad E_5 = \frac{1}{\sqrt{2}}[E_1, E_4], \quad E_6 = \frac{1}{\sqrt{2}}[E_2, E_5]$$

$$H_1 = \widehat{H}_1 + \widehat{H}_2, \quad H_2 = \frac{1}{\sqrt{3}}(\widehat{H}_1 - \widehat{H}_2)$$

$$C_{i,i+1} = H_1^{(i)} \otimes H_1^{(i+1)} + H_2^{(i)} \otimes H_2^{(i+1)} + \sum_{l=1}^{6}\left(E_l^{(i)} \otimes [E_l^{(i+1)}]^\dagger + [E_l^{(i)}]^\dagger \otimes E_l^{(i+1)}\right)$$

$$\widehat{H}_1 = \mathrm{diag}\{1, 0, 1, 0, -1, 0, -1\}, \quad \widehat{H}_2 = \mathrm{diag}\{0, 1, -1, 0, 1, -1, 0\},$$

$$E_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho_0(\lambda) = (1+\lambda)(4+\lambda)(6+\lambda)$$

$$P_{12}^{(0)} = \frac{1}{56}C_{12} - \frac{1}{112}C_{12}^2 - \frac{3}{896}C_{12}^3,$$

$$P_{12}^{(7)} = -\frac{1}{8}C_{12} + \frac{5}{64}C_{12}^2 + \frac{3}{256}C_{12}^3,$$

$$P_{12}^{(14)} = \mathrm{Id}_1 \otimes \mathrm{Id}_2 - \frac{3}{8}C_{12} - \frac{1}{4}C_{12}^2 - \frac{3}{128}C_{12}^3,$$

$$P_{12}^{(27)} = \frac{27}{56}C_{12} + \frac{81}{448}C_{12}^2 + \frac{27}{1792}C_{12}^3$$

$$7 \otimes 7 = 1 \oplus 7 \oplus 14 \oplus 27.$$

$$R_{12}(\lambda) = \rho_0(\lambda)\left(P_{12}^{(27)} + \frac{\lambda-1}{\lambda+1}P_{12}^{(14)} + \frac{\lambda-4}{\lambda+4}P_{12}^{(7)} + \frac{(\lambda-1)(\lambda-6)}{(\lambda+1)(\lambda+6)}P_{12}^{(0)}\right)$$

G2 R and K matrix

$$\text{regularity} \quad : \quad R_{12}(0) = \rho_1(0)^{\frac{1}{2}}\mathcal{P}_{12},$$

$$\text{unitarity} \quad : \quad R_{12}(u)R_{21}(-u) = \rho_1(u) = a(u)a(-u),$$

$$V^i_j = (-1)^{i-1}\delta_{i,\bar{j}}.$$

$$\text{crossing}-\text{symmetry} \quad : \quad R_{12}(u) = -V_1\left\{R_{12}(-u-6)\right\}^{t_2} V_1 = -V_2\left\{R_{12}(-u-6)\right\}^{t_1} V_2,$$

$$R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v).$$

$$K^-(u) = 1 + Mu,$$

$$M = \begin{pmatrix} c_{11} & 0 & 0 & 0 & c_1 & c_2 & 0 \\ 0 & c_{22} & c_3 & 0 & 0 & 0 & -c_2 \\ 0 & c_3 & c_{33} & 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 & c_{33} & -c_3 & 0 \\ c_2 & 0 & 0 & 0 & -c_3 & c_{22} & 0 \\ 0 & -c_2 & c_1 & 0 & 0 & 0 & c_{11} \end{pmatrix},$$

$$c_{11} = \frac{c_1c_3}{c_2} + \frac{c_2c_3}{c_1} - 2,$$

$$c_{22} = 2 - \frac{c_2c_3}{c_1},$$

$$c_{33} = 2 - \frac{c_1c_3}{c_2}.$$

$$\frac{c_1c_3}{c_2} + \frac{c_2c_3}{c_1} + \frac{c_1c_2}{c_3} = 4.$$

Fusion Technique

$$R_{12}(\delta) = P_{12}^{(d)} S_{12} \qquad P_{12}^{(d)}$$ Projector operator

$$\mathrm{V} \otimes \mathrm{V} \to \mathrm{V}^{[\mathrm{d}]}$$

$$P_{12}^{(d)} R_{23}(u) R_{13}(u+\delta) P_{12}^{(d)} = R_{23}(u) R_{13}(u+\delta) P_{12}^{(d)} \Rightarrow R_{[\mathrm{d}]3}(u)$$

$$P_{21}^{(d)} R_{32}(u) R_{31}(u+\delta) P_{21}^{(d)} = R_{32}(u) R_{31}(u+\delta) P_{21}^{(d)} \Rightarrow R_{3[\mathrm{d}]}(u)$$

$$\begin{aligned} &P_{12}^{(d)} K_2^{-}(u) R_{12}(2u+\delta) K_1^{-}(u+\delta) P_{21}^{(d)} \\ &= K_2^{-}(u) R_{12}(2u+\delta) K_1^{-}(u+\delta) P_{21}^{(d)} \end{aligned} \Rightarrow K^{-}_{[\mathrm{d}]}(u)$$

$$\begin{aligned} &P_{21}^{(d)} K_1^{+}(u+\delta) R_{21}(-2u-2\kappa-\delta) K_2^{+}(u) P_{12}^{(d)} \\ &= K_1^{+}(u+\delta) R_{21}(-2u-2\kappa-\delta) K_2^{+}(u) P_{12}^{(d)} \end{aligned} \Rightarrow K^{+}_{[\mathrm{d}]}(u)$$

Projectors

$$R_{12}(u) \sim u^3$$

$$R_{12}(-6) \sim P_{12}^{(1)} \Rightarrow V \otimes V \sim id$$

$$R_{12}(-4) \sim P_{12}^{(7)} \Rightarrow V \otimes V \sim V \Rightarrow R_{23}R_{13} \sim R_{[12]3}$$

$$R_{12}(-1) \sim P_{12}^{(15)} \Rightarrow V \otimes V \sim \bar{V} \Rightarrow R_{23}R_{13} \sim R_{\bar{1}3}$$

$$R_{\bar{1}2}(u) \sim u^2$$

$$R_{\bar{1}2}(-\frac{11}{2}) \sim P_{\bar{1}2}^{(7)} \Rightarrow \bar{V} \otimes V \sim V \Rightarrow R_{23}R_{\bar{1}3} \sim R_{[\bar{1}2]3}$$

$$R_{\bar{1}2}(-\frac{7}{2}) \sim P_{\bar{1}2}^{(34)} \Rightarrow \bar{V} \otimes V \sim \tilde{V} \Rightarrow R_{23}R_{\bar{1}3} \sim R_{\tilde{1}3}$$

$$R_{\tilde{1}2}(u) \sim u^4$$

$$R_{\tilde{1}2}(-\frac{7}{2}) \sim P_{\tilde{1}2}^{(15)} \Rightarrow \tilde{V} \otimes V_2 \sim \bar{V} \Rightarrow R_{23}R_{\tilde{1}3} \sim R_{\bar{1}3}$$

$$R_{\tilde{1}2}(-\frac{13}{2}) \sim P_{\tilde{1}2}^{(7)} \Rightarrow \tilde{V} \otimes V \sim V \Rightarrow R_{23}R_{\tilde{1}3} \sim R_{13}$$

$$R_{\tilde{1}2}(-\frac{9}{2}) \sim P_{\tilde{1}2}^{(49)} \Rightarrow \tilde{V} \otimes V \sim V \otimes V \Rightarrow R_{23}R_{\tilde{1}3} \sim R_{13}R_{23}$$

$$R_{\tilde{1}2}(-\frac{3}{2}) \sim P_{\tilde{1}2}^{(105)} \Rightarrow \tilde{V} \otimes V \sim V \otimes \bar{V} \Rightarrow R_{23}R_{\tilde{1}3} \sim R_{13}R_{\bar{2}3}$$

03

The generation of projection operators

$$T_0(u) = R_{01}(u-\theta_1)R_{02}(u-\theta_2)\cdots R_{0N}(u-\theta_N), \qquad \hat{T}_0(u) = R_{N0}(u+\theta_N)\cdots R_{20}(u+\theta_2)R_{10}(u+\theta_1),$$

$$\begin{aligned}
T_a(\theta_j)T_b(\theta_j+\delta) &= R_{a1}(\theta_j-\theta_1)\cdots R_{aj-1}(\theta_j-\theta_{j-1})R_{aj}(0)R_{aj+1}(\theta_j-\theta_{j+1})\cdots \\
&\quad \times R_{aN}(\theta_j-\theta_N)R_{b1}(\theta_j-\theta_1+\delta)\cdots R_{bj-1}(\theta_j-\theta_{j-1}+\delta)R_{bj}(\delta) \\
&\quad \times R_{aj}(0)R_{ja}(0)\rho_{ab}(0)^{-1}R_{bj+1}(\theta_j-\theta_{j+1}+\delta)\cdots R_{bN}(\theta_j-\theta_N+\delta) \\
&= R_{jj+1}(\theta_j-\theta_{j+1})\cdots R_{jN}(\theta_j-\theta_N)R_{a1}(\theta_j-\theta_1)\cdots R_{aj-1}(\theta_j-\theta_{j-1}) \\
&\quad \times R_{b1}(\theta_j-\theta_1+\delta)\cdots R_{bj-1}(\theta_j-\theta_{j-1}+\delta) \\
&\quad \times P^{(d)}_{ba}S_dR_{ja}(0)R_{bj+1}(\theta_j-\theta_{j+1}+\delta)\cdots R_{bN}(\theta_j-\theta_N+\delta) \\
&= P^{(d)}_{ba}R_{a1}(\theta_j-\theta_1)\cdots R_{aj-1}(\theta_j-\theta_{j-1})R_{aj}(0)R_{ja}(0)\rho_{ab}(0)^{-1}R_{jj+1}(\theta_j-\theta_{j+1})\cdots \\
&\quad \times R_{jN}(\theta_j-\theta_N)R_{b1}(\theta_j-\theta_1+\delta)\cdots R_{bj-1}(\theta_j-\theta_{j-1}+\delta) \\
&\quad \times R_{ba}(\delta)R_{ja}(0)R_{bj+1}(\theta_j-\theta_{j+1}+\delta)\cdots R_{bN}(\theta_j-\theta_N+\delta) \\
&= P^{(d)}_{ba}T_a(\theta_j)T_b(\theta_j+\delta).
\end{aligned}$$

$$\hat{T}_a(-\theta_j)\hat{T}_b(-\theta_j+\delta) = P^{(d)}_{ab}\hat{T}_a(-\theta_j)\hat{T}_b(-\theta_j+\delta),$$

Operator identity

$$t^{(p)}(u)=tr_0T_0(u),\quad \bar{t}^{(p)}(u)=tr_{\bar{0}}T_{\bar{0}}(u),\quad \tilde{t}^{(p)}(u)=tr_{\tilde{0}}T_{\tilde{0}}(u).$$

(1) $$t^{(p)}(\theta_j)\,t^{(p)}(\theta_j-6)=\prod_{i=1}^{N}a(\theta_j-\theta_i)e(\theta_j-\theta_i-6)\times \mathrm{id},$$

(2) $$t^{(p)}(\theta_j)\,t^{(p)}(\theta_j-4)=\prod_{i=1}^{N}(\theta_j-\theta_i+1)(\theta_j-\theta_i-4)(\theta_j-\theta_i-6)\,t^{(p)}(\theta_j-2),$$

(3) $$t^{(p)}(\theta_j)\,t^{(p)}(\theta_j-1)=\prod_{i=1}^{N}(\theta_j-\theta_i-1)a(\theta_j-\theta_i)\,\bar{t}^{(p)}(\theta_j-\frac{1}{2}),$$

(4) $$t^{(p)}(\theta_j)\,\bar{t}^{(p)}(\theta_j-\frac{11}{2})=\prod_{i=1}^{N}(\theta_j-\theta_i+4)(\theta_j-\theta_i+6)\,t^{(p)}(\theta_j-5),$$

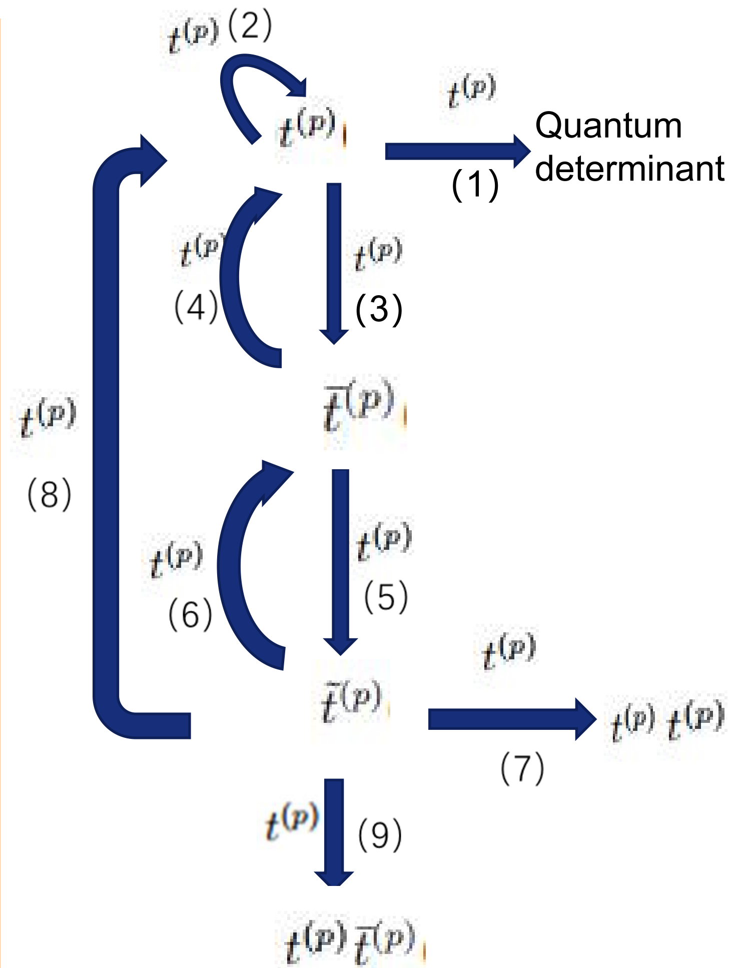
(5) $$t^{(p)}(\theta_j)\,\bar{t}^{(p)}(\theta_j-\frac{7}{2})=\prod_{i=1}^{N}(\theta_j-\theta_i+6)\,\tilde{t}^{(p)}(\theta_j-\frac{5}{2}),$$

(6) $$t^{(p)}(\theta_j)\,\tilde{t}^{(p)}(\theta_j-\frac{7}{2})=\prod_{i=1}^{N}(\theta_j-\theta_i-1)(\theta_j-\theta_i-4)a(\theta_j-\theta_i)\,\bar{t}^{(p)}(\theta_j-\frac{5}{2}),$$

(7) $$t^{(p)}(\theta_j)\,\tilde{t}^{(p)}(\theta_j-\frac{9}{2})=\prod_{i=1}^{N}(\theta_j-\theta_i+6)\,t^{(p)}(\theta_j-2)t^{(p)}(\theta_j-5),$$

(8) $$t^{(p)}(\theta_j)\,\tilde{t}^{(p)}(\theta_j-\frac{13}{2})=\prod_{i=1}^{N}(\theta_j-\theta_i-4)a(\theta_j-\theta_i)\,t^{(p)}(\theta_j-7),$$

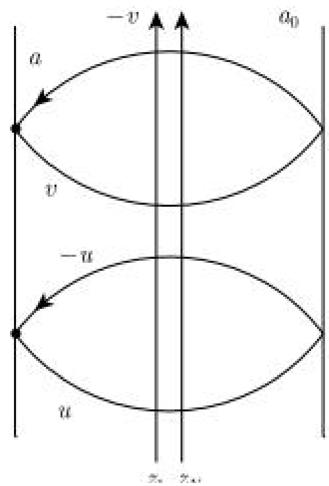
(9) $$t^{(p)}(\theta_j)\,\tilde{t}^{(p)}(\theta_j-\frac{3}{2})=\prod_{i=1}^{N}(\theta_j-\theta_i-1)(\theta_j-\theta_i-6)\,t^{(p)}(\theta_j-2)\bar{t}^{(p)}(\theta_j-\frac{1}{2}),$$

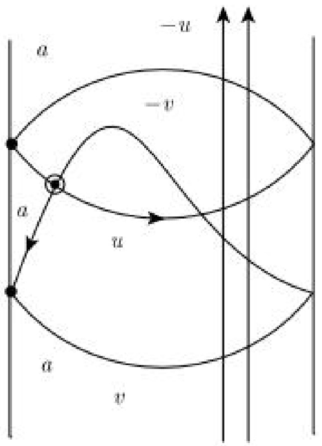


$$t^{(p)}(u)|_{u\to\pm\infty}=7u^{3N}\times \mathrm{id}+\cdots,\quad \bar{t}^{(p)}(u)|_{u\to\pm\infty}=15u^{2N}\times \mathrm{id}+\cdots,$$

$$\tilde{t}^{(p)}(u)|_{u\to\pm\infty}=34u^{4N}\times \mathrm{id}+\cdots.$$

Fusion for open boundary





$$
\begin{aligned}
t_a(u)t_b(u+\delta) &= tr_a\{K_a^+(u)T_a(u)K_a^-(u)\hat{T}_a(u)\}\\
&\quad \times tr_b\{K_b^+(u+\delta)T_b(u+\delta)K_b^-(u+\delta)\hat{T}_b(u+\delta)\}^{t_b}\\
&= tr_{ab}\{K_a^+(u)T_a(u)K_a^-(u)\hat{T}_a(u)[T_b(u+\delta)K_b^-(u+\delta)\hat{T}_b(u+\delta)]^{t_b}[K_b^+(u+\delta)]^{t_b}\}\\
&= [\tilde{\rho}_{ab}(2u+\delta)]^{-1}tr_{ab}\{K_a^+(u)T_a(u)K_a^-(u)\hat{T}_a(u)[T_b(u+\delta)K_b^-(u+\delta)\\
&\quad \times \hat{T}_b(u+\delta)]^{t_b}R_{ba}^{t_b}(2u+\delta)R_{ab}^{t_b}(-2u-2\kappa-\delta)[K_b^+(u+\delta)]^{t_b}\}\\
&= [\tilde{\rho}_{ab}(2u+\delta)]^{-1}tr_{ab}\{[K_b^+(u+\delta)R_{ab}(-2u-2\kappa-\delta)K_a^+(u)T_a(u)\\
&\quad \times K_a^-(u)\hat{T}_a(u)]^{t_b}[R_{ba}(2u+\delta)T_b(u+\delta)K_b^-(u+\delta)\hat{T}_b(u+\delta)]^{t_b}\}\\
&= [\tilde{\rho}_{ab}(2u+\delta)]^{-1}tr_{ab}\{K_b^+(u+\delta)R_{ab}(-2u-2\kappa-\delta)K_a^+(u)T_a(u)\\
&\quad \times K_a^-(u)\hat{T}_a(u)R_{ba}(2u+\delta)T_b(u+\delta)K_b^-(u+\delta)\hat{T}_b(u+\delta)\}\\
&= [\tilde{\rho}_{ab}(2u+\delta)]^{-1}tr_{ab}\{K_b^+(u+\delta)R_{ab}(-2u-2\kappa-\delta)K_a^+(u)T_a(u)T_b(u+\delta)\\
&\quad \times K_a^-(u)R_{ba}(2u+\delta)K_b^-(u+\delta)\hat{T}_a(u)\hat{T}_b(u+\delta)\}.
\end{aligned}
$$

Crossing symmetry for transfer matrix

$$R_{12}(u) = -V_1 R_{21}^{t_1}(-u-6)V_1^{-1} = -V_2^{t_2} R_{21}^{t_2}(-u-6)[V_2^{t_2}]^{-1}$$

$$T_0^{t_0}(-u-6) \ = (-1)^N [V_0^{t_0}]^{-1} \hat{T}_0(u) V_0^{t_0} \qquad \hat{T}_0^{t_0}(-u-6) = (-1)^N V_0^{-1} T_0(u) V_0.$$

$$tr_1\{R_{12}(0)R_{12}(2u)V_1[K_1^-(-u-6)]^{t_1}[V_1^{t_1}]^{-1}\} = f(u)K_2^-(u),$$
$$tr_2\{R_{12}(0)R_{12}(2u)K_2^+(u)\} = f(u)V_1^{t_1}K_1^+(-u-6)^{t_1}V_1^{-1},$$

$$\begin{aligned}
t(-u-6) &= tr_0\{K_0^+(-u-6)T_0(-u-6)\}^{t_0}\{K_0^-(-u-6)\hat{T}_0(-u-6)\}^{t_0} \\
&= tr_0\hat{T}_0(u)V_0^{t_0}\{K_0^+(-u-6)\}^{t_0}V_0^{-1}T_0(u)V_0\{K_0^-(-u-6)\}^{t_0}[V_0^{t_0}]^{-1} \\
&= tr_0\hat{T}_0(u)tr_1R_{01}(0)R_{01}(2u)K_1^+(u)T_0(u)V_0\{K_0^-(-u-6)\}^{t_0}[V_0^{t_0}]^{-1}/f(u) \\
&= tr_1tr_0R_{10}(0)\hat{T}_1(u)R_{01}(2u)T_0(u)V_0\{K_0^-(-u-6)\}^{t_0}[V_0^{t_0}]^{-1}K_1^+(u)/f(u) \\
&= tr_1tr_0R_{10}(0)T_0(u)R_{01}(2u)\hat{T}_1(u)V_0\{K_0^-(-u-6)\}^{t_0}[V_0^{t_0}]^{-1}K_1^+(u)/f(u) \\
&= tr_1T_1(u)tr_0R_{01}R_{01}(2u)V_0\{K_0^-(-u-6)\}^{t_0}[V_0^{t_0}]^{-1}\hat{T}_1(u)K_1^+(u)/f(u) \\
&= tr_1K_1^+(u)T_1(u)K_1^-(u)\hat{T}_1(u) = t(u).
\end{aligned}$$

Operator identity

(1) $$t(\pm\theta_j)\, t(\pm\theta_j - 6) \sim \mathrm{id},$$

(2) $$t(\pm\theta_j)\, t(\pm\theta_j - 4) \sim t(\pm\theta_j - 2),$$

(3) $$t(\pm\theta_j)\, t(\pm\theta_j - 1) \sim \bar{t}^{(p)}(\pm\theta_j - \frac{1}{2}),$$

(4) $$t(\pm\theta_j)\, \bar{t}(\pm\theta_j - \frac{11}{2}) \sim t(\pm\theta_j - 5),$$

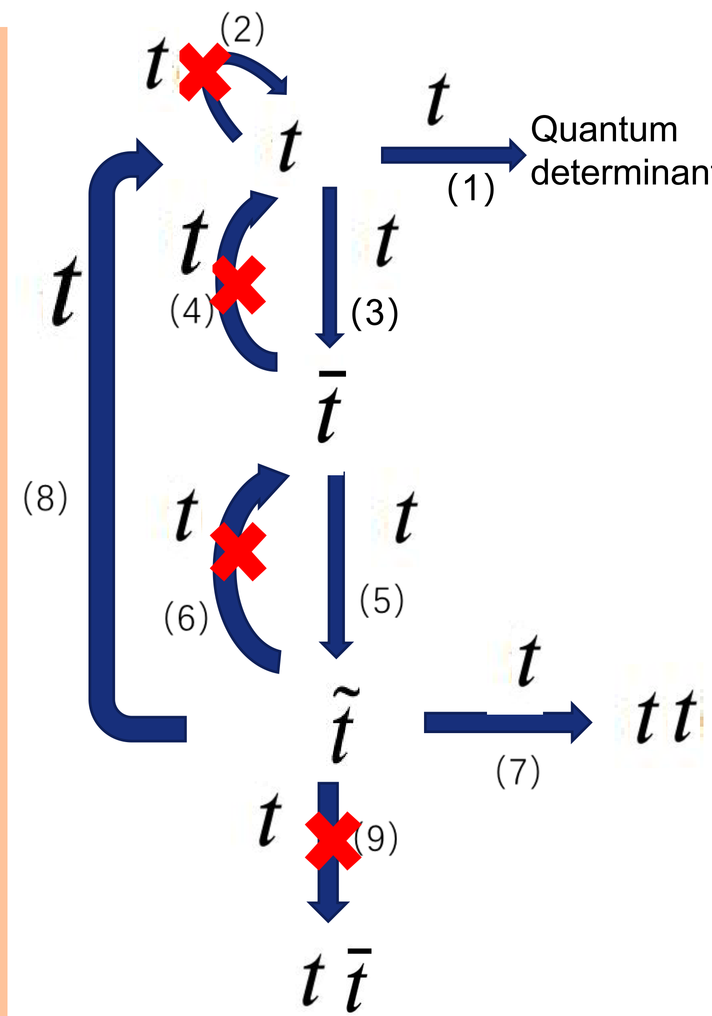
(5) $$t(\pm\theta_j)\, \bar{t}(\pm\theta_j - \frac{7}{2} \sim \tilde{t}(\pm\theta_j - \frac{5}{2}),$$

(6) $$t(\pm\theta_j)\, \tilde{t}(\pm\theta_j - \frac{7}{2}) \sim \bar{t}(\pm\theta_j - \frac{5}{2}),$$

(7) $$t(\pm\theta_j)\, \tilde{t}(\pm\theta_j - \frac{9}{2}) \sim t(\pm\theta_j - 2) t(\pm\theta_j - 5),$$

(8) $$t(\pm\theta_j)\, \tilde{t}(\pm\theta_j - \frac{13}{2}) \sim t(\pm\theta_j - 7),$$

(9) $$t(\pm\theta_j)\, \tilde{t}(\pm\theta_j - \frac{3}{2}) \sim t(\pm\theta_j - 2)\bar{t}(\pm\theta_j - \frac{1}{2}),$$



03

Function relations

$$t(-u-6)=t(u).$$

$$\bar{t}(u)=\bar{t}(-u-6).$$

$$\tilde{t}(u)=\tilde{t}(-u-6).$$

$$\Lambda(\theta_j)\Lambda(-\theta_j)=4^2\frac{(\theta_j-1)(-\theta_j)(\theta_j+1)(\theta_j+6)}{(\theta_j-2)(\theta_j-3)(\theta_j+2)(\theta_j+3)}$$

$$\times(\theta_j-\frac{1}{2})(\theta_j-\frac{5}{2})(\theta_j+\frac{1}{2})(\theta_j+\frac{5}{2})\prod_{i=1}^{N}\rho_{12}(\theta_j-\theta_i)\rho_{12}(\theta_j+\theta_i),$$

$$\Lambda(\pm\theta_j)\Lambda(\pm\theta_j-1)=-\frac{(\pm\theta_j-1)(\pm\theta_j+6)(\pm\theta_j+\frac{5}{2})^2}{(\pm\theta_j+2)(\pm\theta_j+3)}$$

$$\times\prod_{i=1}^{N}(\pm\theta_j-\theta_i-1)(\pm\theta_j+\theta_i-1)a(\pm\theta_j-\theta_i)a(\pm\theta_j+\theta_i)\bar{\Lambda}(\pm\theta_j-\frac{1}{2}),$$

03

Function relations

$$\Lambda(\pm\theta_j)\bar{\Lambda}(\pm\theta_j-\frac{7}{2})=-\frac{(\pm\theta_j+1)(\pm\theta_j+6)}{(\pm\theta_j-\frac{1}{2})(\pm\theta_j-\frac{3}{2})(\pm\theta_j+3)(\pm\theta_j+4)}$$
$$\times\prod_{i=1}^{N}(\pm\theta_j-\theta_i+6)(\pm\theta_j+\theta_i+6)\tilde{\Lambda}(\pm\theta_j-\frac{5}{2}),$$

$$\Lambda(\pm\theta_j)\tilde{\Lambda}(\pm\theta_j-\frac{9}{2})=2^4\frac{(\pm\theta_j+1)(\pm\theta_j+6)}{(\pm\theta_j+3)(\pm\theta_j+4)}(\pm\theta_j-\frac{1}{2})(\pm\theta_j-\frac{7}{2})(\pm\theta_j+\frac{1}{2})(\pm\theta_j+\frac{5}{2})$$
$$\times\prod_{i=1}^{N}(\pm\theta_j-\theta_i+6)(\pm\theta_j+\theta_i+6)\Lambda(\pm\theta_j-2)\Lambda(\pm\theta_j-5). \quad (3.4)$$

$$\Lambda(\pm\theta_j)\tilde{\Lambda}(\pm\theta_j-\frac{13}{2})=-2^6\frac{(\pm\theta_j-4)(\pm\theta_j+1)(\pm\theta_j+6)}{(\pm\theta_j-2)(\pm\theta_j+2)(\pm\theta_j+3)}$$
$$\times(\pm\theta_j-\frac{11}{2})(\pm\theta_j-\frac{5}{2})(\pm\theta_j-\frac{3}{2})(\pm\theta_j-\frac{1}{2})(\pm\theta_j+\frac{1}{2})(\pm\theta_j+\frac{5}{2})$$
$$\times\prod_{i=1}^{N}(\pm\theta_j-\theta_i+4)(\pm\theta_j+\theta_i+4)a(\pm\theta_j-\theta_i)a(\pm\theta_j+\theta_i)\Lambda(\pm\theta_j-7).$$

Special points and asymptotic behaviors

$$\Lambda(u)|_{u\to\pm\infty} = Au^{6N+2} + \cdots, \quad A = -2[-2 + 2(c_1\tilde{c}_1 + c_2\tilde{c}_2 + c_3\tilde{c}_3) + \frac{c_1c_2\tilde{c}_1\tilde{c}_2}{c_3\tilde{c}_3} + \frac{c_1c_3\tilde{c}_1\tilde{c}_3}{c_2\tilde{c}_2} + \frac{c_3c_2\tilde{c}_3\tilde{c}_2}{c_1\tilde{c}_1}].$$

$$\bar{\Lambda}(u)|_{u\to\pm\infty} = 16[-(\frac{A}{8})^2 + \frac{A}{8} + \frac{3}{4}]u^{4N+2} + \cdots,$$

$$\tilde{\Lambda}(u)|_{u\to\pm\infty} = -128[\frac{3}{2}(\frac{A}{8})^2 + \frac{A}{16} + \frac{3}{8}]u^{8N+6} + \cdots,$$

3N+2N+4N+1+1+
1+1+3+1=9N+8

$$\Lambda(0) = -5\prod_{l=1}^{N}\rho_{12}(\theta_l),$$

$$\Lambda(-1) = -\frac{5}{4}\prod_{l=1}^{N}(\theta_l - 1)(-\theta_l - 1)\bar{\Lambda}(-\frac{1}{2}),$$

$$\tilde{\Lambda}(-\frac{5}{2}) = -\frac{15}{2}\prod_{l=1}^{N}(\theta_l + 1)(-\theta_l + 1)(\theta_l + 4)(-\theta_l + 4)\bar{\Lambda}(-\frac{7}{2}),$$

$$\tilde{\Lambda}(-\frac{13}{2}) = 330\prod_{l=1}^{N}(\theta_l - 4)(-\theta_l - 4)\Lambda(-7),$$

$$\tilde{\Lambda}(-1) = 0.$$

$$P^{(7)\perp}_{\bar{1}2} = 1 - P^{(7)}_{\bar{1}2},$$

$$\begin{aligned} t(u)\tilde{t}(u - \frac{13}{2}) &= -2^6\frac{(u-4)(u+1)(u+6)}{(u-2)(u+2)(u+3)} \\ &\times(u - \frac{11}{2})(u - \frac{5}{2})(u - \frac{3}{2})(u - \frac{1}{2})(u + \frac{1}{2})(u + \frac{5}{2}) \\ &\times\prod_{i=1}^{N}(u - \theta_i + 4)(u + \theta_i + 4)a(u - \theta_i)a(u + \theta_i)t(u - 7) \\ &+\frac{u(u - \frac{11}{2})(u - \frac{3}{2})}{2^8(u+3)(u+2)(u+\frac{3}{2})(u+\frac{1}{2})(u-1)(u-2)(u-\frac{5}{2})(u-\frac{7}{2})} \\ &\times\prod_{i=1}^{N}(u - \theta_i)(u + \theta_i)\tilde{t}^{\perp}(u - 7). \end{aligned}$$

T - Q relations

$$\Lambda(u) = Z_1(u) + Z_2(u) + Z_3(u) + Z_4(u) + Z_5(u) + Z_6(u) + Z_7(u) + f_1(u) + f_2(u),$$

$$Z_1(u) = -4\frac{(u+1)(u+6)}{(u+2)(u+3)}(u+\frac{1}{2})(u+\frac{5}{2})\prod_{j=1}^{N} a(u-\theta_j)a(u+\theta_j)\frac{Q^{(1)}(u-1)}{Q^{(1)}(u)}$$

$$Z_2(u) = -4\frac{u(u+6)}{(u+2)(u+3)}(u+\frac{1}{2})(u+\frac{5}{2})\prod_{j=1}^{N} b(u-\theta_j)b(u+\theta_j)\frac{Q^{(1)}(u+1)Q^{(2)}(u-3)}{Q^{(1)}(u)Q^{(2)}(u)}$$

$$Z_3(u) = -4\frac{u(u+6)}{(u+2)(u+3)}(u+\frac{7}{2})(u+\frac{5}{2})\prod_{j=1}^{N} b(u-\theta_j)b(u+\theta_j)\frac{Q^{(1)}(u+1)Q^{(2)}(u+3)}{Q^{(1)}(u+3)Q^{(2)}(u)}$$

$$Z_4(u) = -4\frac{u(u+6)}{(u+2)(u+4)}(u+\frac{7}{2})(u+\frac{5}{2})\prod_{j=1}^{N} c(u-\theta_j)c(u+\theta_j)\frac{Q^{(1)}(u+1)Q^{(1)}(u+4)}{Q^{(1)}(u+2)Q^{(1)}(u+3)}$$

$$Z_5(u) = -4\frac{u(u+6)}{(u+3)(u+4)}(u+\frac{7}{2})(u+\frac{5}{2})\prod_{j=1}^{N} d(u-\theta_j)d(u+\theta_j)\frac{Q^{(1)}(u+4)Q^{(2)}(u-1)}{Q^{(1)}(u+2)Q^{(2)}(u+2)}$$

$$Z_6(u) = -4\frac{u(u+6)}{(u+3)(u+4)}(u+\frac{7}{2})(u+\frac{11}{2})\prod_{j=1}^{N} d(u-\theta_j)d(u+\theta_j)\frac{Q^{(1)}(u+4)Q^{(2)}(u+5)}{Q^{(1)}(u+5)Q^{(2)}(u+2)}$$

$$Z_7(u) = -4\frac{u(u+5)}{(u+3)(u+4)}(u+\frac{7}{2})(u+\frac{11}{2})\prod_{j=1}^{N} e(u-\theta_j)e(u+\theta_j)\frac{Q^{(1)}(u+6)}{Q^{(1)}(u+5)}$$

03

T - Q relations

$$Q^{(1)}(u)=\prod_{k=1}^{L_1}(iu+\mu_k^{(1)}+i\frac{1}{2})(iu-\mu_k^{(1)}+i\frac{1}{2}),$$

$$Q^{(2)}(u)=\prod_{k=1}^{L_2}(iu+\mu_k^{(2)}+2i)(iu-\mu_k^{(2)}+2i).$$

$$f_1(u)=-4\frac{u(u+6)}{(u+3)}(u+\frac{1}{2})(u+\frac{5}{2})(u+\frac{7}{2})\times\prod_{j=1}^{N}b(u-\theta_j)b(u+\theta_j)\frac{Q^{(1)}(u+1)}{Q^{(2)}(u)}x,$$

$$f_2(u)=-4\frac{u(u+6)}{(u+3)}(u+\frac{5}{2})(u+\frac{7}{2})(u+\frac{11}{2})\times\prod_{j=1}^{N}d(u-\theta_j)d(u+\theta_j)\frac{Q^{(1)}(u+4)}{Q^{(2)}(u+2)}x,$$

BA equations

$$\frac{Q^{(1)}(i\mu_k^{(1)}+\frac{1}{2})Q^{(2)}(i\mu_k^{(1)}-\frac{7}{2})}{Q^{(1)}(i\mu_k^{(1)}-\frac{3}{2})Q^{(2)}(i\mu_k^{(1)}-\frac{1}{2})}=-\frac{(i\mu_k^{(1)}+\frac{1}{2})}{(i\mu_k^{(1)}-\frac{1}{2})}\frac{\prod_{j=1}^{N}(i\mu_k^{(1)}-\theta_j+\frac{1}{2})(i\mu_k^{(1)}+\theta_j+\frac{1}{2})}{\prod_{j=1}^{N}(i\mu_k^{(1)}-\theta_j-\frac{1}{2})(i\mu_k^{(1)}+\theta_j-\frac{1}{2})},$$
$$k=1,2,\cdots,L_1,$$

$$\frac{(i\mu_l^{(2)}-\frac{3}{2})}{(i\mu_l^{(2)})}\frac{Q^{(2)}(i\mu_l^{(2)}-5)}{Q^{(1)}(i\mu_l^{(2)}-2)}+\frac{(i\mu_l^{(2)}+\frac{3}{2})}{(i\mu_l^{(2)})}\frac{Q^{(2)}(i\mu_l^{(2)}+1)}{Q^{(1)}(i\mu_l^{(2)}+1)}=-x(i\mu_l^{(2)}-\frac{3}{2})(i\mu_l^{(2)}+\frac{3}{2}),\quad l=1,2,\cdots,L_2.$$

$$L_2=L_1,$$

$$x=\frac{1}{4}[-16+2(c_1\tilde{c}_1+c_2\tilde{c}_2+c_3\tilde{c}_3)+\frac{c_1c_2\tilde{c}_1\tilde{c}_2}{c_3\tilde{c}_3}+\frac{c_1c_3\tilde{c}_1\tilde{c}_3}{c_2\tilde{c}_2}+\frac{c_3c_2\tilde{c}_3\tilde{c}_2}{c_1\tilde{c}_1}].$$

Numerical solutions for small N has been done for BA equations

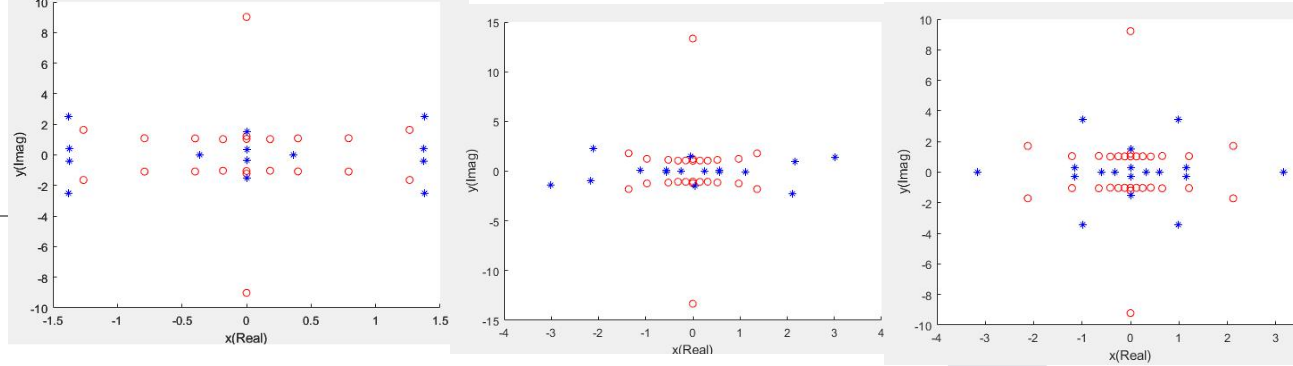
T - Q relations

$$
\begin{aligned}
\bar{\Lambda}(u-\frac{1}{2}) = & -\frac{(u+2)(u+3)}{(u-1)(u+6)(u+\frac{5}{2})(u+\frac{5}{2})} \\
& \times[\prod_{i=1}^{N}(u+\theta_i-1)(u-\theta_i-1)a(u-\theta_i)a(u+\theta_i)]^{-1} \\
& \times\{Z_1(u)[\sum_{i=2}^{7} Z_i(u-1)+f_1(u-1)+f_2(u-1)] \\
& +[\sum_{i=2}^{6} Z_i(u)+f_1(u)+f_2(u)]Z_7(u-1) \\
& +(Z_2(u)+f_1(u)+Z_3(u))(Z_5(u-1)+f_2(u-1)+Z_6(u-1))\},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Lambda}(u-\frac{5}{2}) = & \frac{u(u-1)(u+4)(u-\frac{1}{2})(u-\frac{3}{2})}{(u+1)(u+6)(u-4)(u-\frac{1}{2})^2} \\
& \times[\prod_{i=1}^{N}(u+\theta_i-4)(u-\theta_i-4)(u+\theta_i+6)(u-\theta_i+6)a(u+\theta_i-3)a(u-\theta_i-3)]^{-1} \\
& \times\{(\sum_{i=1}^{4} Z_i(u)+f_1(u))(\sum_{k=1}^{6} Z_k(u-3)+f_1(u-3)+f_2(u-3))Z_7(u-4) \\
& +Z_1(u)Z_1(u-3)(\sum_{k=4}^{6} Z_k(u-4)+f_2(u-4))+Z_5(u)Z_6(u-3)Z_7(u-4) \\
& +Z_1(u)(Z_2(u-3)+f_1(u-3)+Z_3(u-3))(Z_5(u-4)+f_2(u-4)+Z_6(u-4)) \\
& +Z_1(u)Z_3(u-3)(Z_5(u-4)+f_2(u-4)+Z_6(u-4))\}.
\end{aligned}
$$

04 Discussions

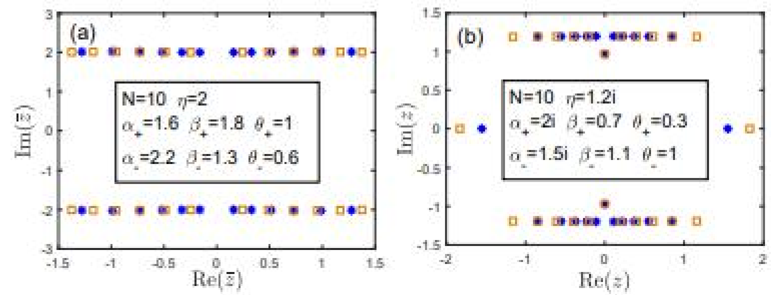
04 Discussions



Problems:

1. For high rank B_n, D_n ,the number of operator identity is rather large ;
2. Is it applicable for other exceptional Lie algebras? $D_4^{(3)}, E_6^{(1)}, E_6^{(2)}$
3. How to carry the method on the super Lie algebras?

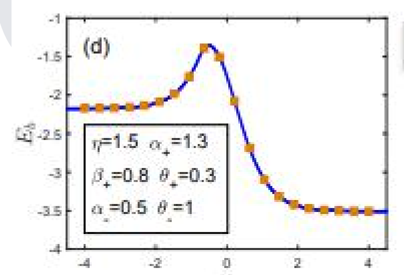
$$\Lambda(u) = \Lambda_0 \prod_{j=1}^{N+2} \sinh(u - z_j + \frac{\eta}{2}) \sinh(u + z_j + \frac{\eta}{2}).$$



Future work:

1. Searching for the rule of the distribution for Bethe roots and zero roots (T-W) ;
2. Calculating thermodynamic quantities and considering their possible applications.

Y. Qiao, J. Cao, W.-L. Yang, K. Shi and Y. Wang, Phys. Rev. B 103(2021), L220401



Thank you!