Ocneanu Algebra of Defect Seams: Critical A-D-E RSOS Lattice Models

Wuhan, 21 Nov 2025

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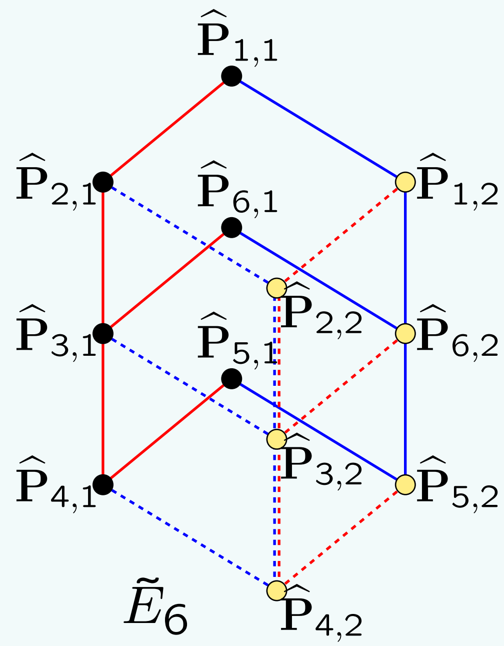

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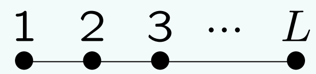
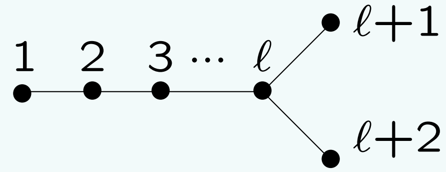
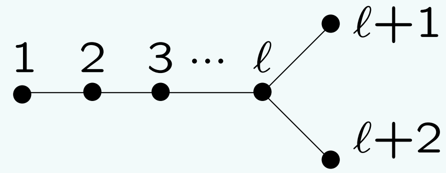
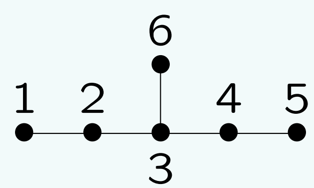
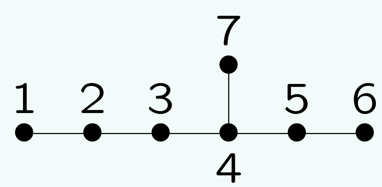
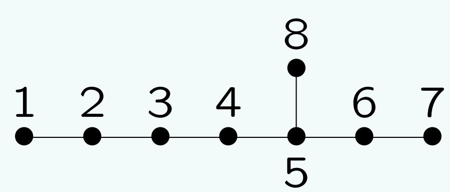
Simply Laced A-D-E Lie Algebras: Dynkin Diagrams

	Graph G	g	Exp(G)	Type/H	Γ
A_L		$L+1$	$1, 2, \cdots, L$	I	$\mathbb{Z}_2$
$D_{\ell+2}$ (ℓ odd)		$2\ell+2$	$1, 3, \cdots, 2\ell+1, \ell+1$	II/$A_{2\ell+1}$	$\mathbb{Z}_2$
$D_{\ell+2}$ (ℓ even)		$2\ell+2$	$1, 3, \cdots, 2\ell+1, \ell+1$	I	$\mathbb{Z}_2/\mathbb{S}_3$
E_6		12	$1, 4, 5, 7, 8, 11$	I	$\mathbb{Z}_2$
E_7		18	$1, 5, 7, 9, 11, 13, 17$	II/D_{10}	1
E_8		30	$1, 7, 11, 13, 17, 19, 23, 29$	I	1

- A-D-E Dynkin diagrams G, Coxeter numbers g, Coxeter exponents Exp(G), the type I or II, parent graphs $H \neq G$ and diagram automorphism group Γ. Nodes 1 = identity, 2 = fundamental.
- The eigenvalues of the adjacency matrix G are $2\cos m\lambda$ with $\lambda = \frac{\pi}{g}$ and $m \in$ Exp(G).

Fusion Matrices (nimreps)

Verlinde Matrices: $G = A_L$ ($L \times L$ Matrices): $N_i = (N_i)_j{}^k, \quad i,j,k \in A_L$

$$N_1 = I, \quad N_2 = A_L, \quad N_j = N_{j-1}N_2 - N_{j-2}, \quad N_L = \sigma \in \mathbb{Z}_2$$

$$\boxed{N_i\, N_j = \sum_{k\in A_L} N_{ij}{}^k\, N_k, \quad L = g-1, \qquad N_j = U_{j-1}(\tfrac{1}{2}N_2) = \text{Chebyshev polynomials}}$$

Tricritical Ising Model: $G = A_4, \quad g = 5$

$$N_1 = I, \quad N_2 = \begin{pmatrix} 0&1&0&0\\ 1&0&1&0\\ 0&1&0&1\\ 0&0&1&0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0&0&1&0\\ 0&1&0&1\\ 1&0&1&0\\ 0&1&0&0 \end{pmatrix}, \quad N_4 = \begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{pmatrix} = \sigma$$

Fused Adjacency/Intertwiner Matrices:

$$|G| \times |G| \text{ Matrices:} \qquad n_i = (n_i)_a{}^b, \quad i = 1,2,\ldots,g-1; \quad a,b \in G$$

$$n_1 = I, \quad n_2 = G, \quad n_j = n_2 n_{j-1} - n_{j-2}, \quad 3 \le j \le g-1$$

$$\boxed{n_i\, n_j = \sum_{k\in A_{g-1}} N_{ij}{}^k\, n_k, \quad n_j = U_{j-1}(\tfrac{1}{2}n_2)} \qquad n_{g-1} = \begin{cases} I, & D_{2\ell}, E_7, E_8 \\ \sigma \in \mathbb{Z}_2, & A_L, D_{2\ell-1}, E_6 \end{cases}$$

Critical 3-State Potts Model: $G = D_4, \quad g = 6$

$$n_1 = n_5 = I, \quad n_2 = n_4 = \begin{pmatrix} 0&1&0&0\\ 1&0&1&1\\ 0&1&0&0\\ 0&1&0&0 \end{pmatrix}, \quad n_3 = \begin{pmatrix} 0&0&1&1\\ 0&2&0&0\\ 1&0&0&1\\ 1&0&1&0 \end{pmatrix}$$

Graph Fusion Matrices (nimreps)

Graph Fusion Matrices: Entries of $\hat{N}_a$ are non-negative integers only for Type I models:

$$|G|\times|G| \text{ Matrices:}\quad \hat{N}_a=(\hat{N}_a)_b{}^c,\quad \hat{N}_1=I,\quad \hat{N}_2=G,\quad \hat{N}_2\hat{N}_a=\sum_{b\in G} G_{ab}\,\hat{N}_b,\quad a,b,c\in G$$

$$\boxed{\hat{N}_a\hat{N}_b=\sum_{c\in G}\hat{N}_{ab}{}^c\,\hat{N}_c,\qquad n_i\hat{N}_a=\sum_{b\in G} n_{ia}{}^b\hat{N}_b,\qquad n_i=\sum_{a\in G} n_{i1}{}^a\hat{N}_a}$$

For E_6, the rectangular fundamental intertwiner $n_{i1}{}^a$ admits a generalized left inverse giving

$$\boxed{\hat{N}_a=n_a,\quad a=1,2,3;\qquad \hat{N}_4=n_6-n_2,\quad \hat{N}_5=n_5-n_3,\quad \hat{N}_6=n_2+n_4-n_6}$$

Critical Ising: $G=A_3,\quad g=4$ **Fusion Rules:** $\sigma^2=I+\varepsilon,\quad \sigma\varepsilon=\varepsilon\sigma=\sigma,\quad \varepsilon^2=I$

$$\hat{N}_1=N_1=I,\quad \hat{N}_2=N_2=\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}=\sigma,\quad \hat{N}_3=N_3=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}=\varepsilon$$

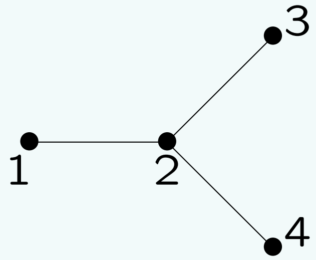
Critical 3-State Potts: $G=D_4,\quad g=6,\quad \omega\in\mathbb{Z}_3$

$$\hat{N}_1=I,\quad \hat{N}_2=N_2=\begin{pmatrix}0&1&0&0\\1&0&1&1\\0&1&0&0\\0&1&0&0\end{pmatrix}=G,\quad \hat{N}_3=\begin{pmatrix}0&0&1&0\\0&1&0&0\\0&0&0&1\\1&0&0&0\end{pmatrix}=\omega,\quad \hat{N}_4=\begin{pmatrix}0&0&0&1\\0&1&0&0\\1&0&0&0\\0&0&1&0\end{pmatrix}=\omega^2$$

Cayley Table/Fundamental Intertwiner

D_4 :

	I	G	ω	ω^2
I	I	G	ω	ω^2
G	G	$I+\omega+\omega^2$	G	G
ω	ω	G	ω^2	I
ω^2	ω^2	G	I	ω



$$\sigma=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\qquad C=(n_i)_1{}^a=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&1&0&0\\1&0&0&0\end{pmatrix}$$

Critical A-D-E Lattice Models

- The face weights of the critical (unitary) A-D-E lattice models [ABF84,Pasquier87] are

$$\boxed{W\!\begin{pmatrix} d & c \\ a & b \end{pmatrix}\!\!\Big|\, u\Big) = \begin{array}{ccc} d & & c \\ & \boxed{u} & \\ a & & b \end{array} = \frac{\sin(\lambda-u)}{\sin\lambda}\,\delta_{ac} + \frac{\sin u}{\sin\lambda}\sqrt{\frac{\psi_a\psi_c}{\psi_b\psi_d}}\,\delta_{bd}, \qquad \lambda=\frac{\pi}{g}, \qquad 0\le u\le\lambda}$$

where u is the spectral parameter. The face weights vanish if $G_{ab}G_{bc}G_{cd}G_{da}=0$ and satisfy the Yang-Baxter equation implying commuting transfer matrices and integrability.

- The largest eigenvalue of the adjacency matrix G is $[2]_x = 2\cos\lambda$ with eigenvectors ψ

$$G\psi = [2]_x\,\psi, \qquad \psi = (\psi_a)_{1\le a\le |G|} = \begin{cases} \Big([1]_x, [2]_x, \ldots, [L]_x\Big), & G = A_L \\ \Big([1]_x, [2]_x, \ldots, [\ell]_x, \frac{[\ell]_x}{[2]_x}, \frac{[\ell]_x}{[2]_x}\Big), & G = D_{\ell+2} \\ \Big([1]_x, [2]_x, [3]_x, [2]_x, [1]_x, \frac{[3]_x}{[2]_x}\Big), & G = E_6 \\ \Big([1]_x, [2]_x, [3]_x, [4]_x, \frac{[6]_x}{[2]_x}, \frac{[4]_x}{[3]_x}, \frac{[4]_x}{[2]_x}\Big), & G = E_7 \\ \Big([1]_x, [2]_x, [3]_x, [4]_x, [5]_x, \frac{[7]_x}{[2]_x}, \frac{[5]_x}{[3]_x}, \frac{[5]_x}{[2]_x}\Big), & G = E_8 \end{cases}$$

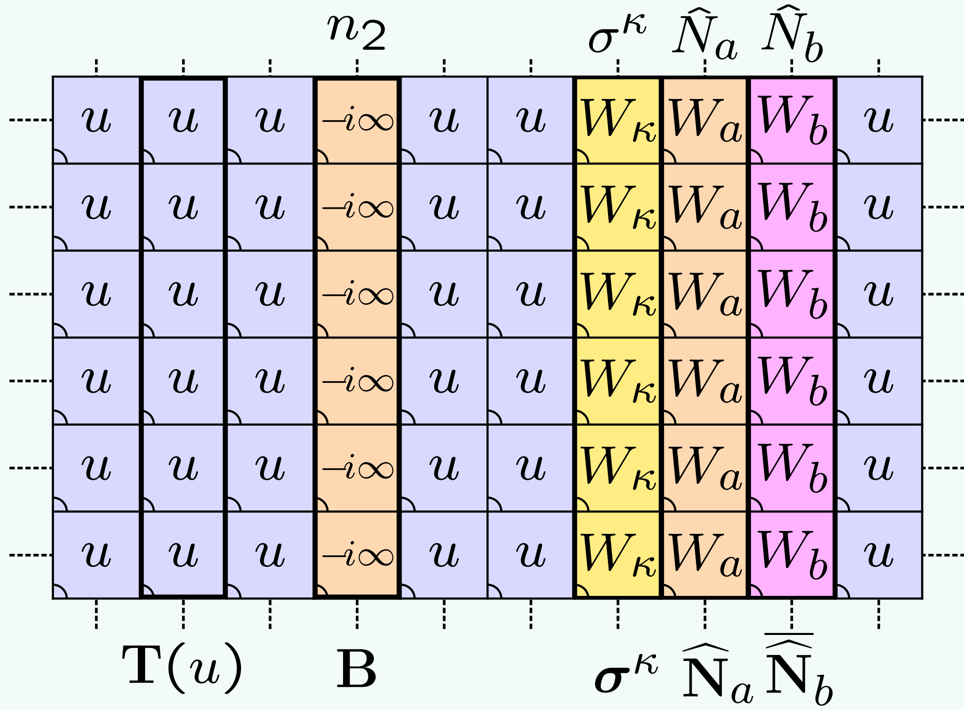
Here the quantum dimensions are $[a]_x = \frac{x^a - x^{-a}}{x - x^{-1}} = \frac{\sin a\lambda}{\sin\lambda}$ with $x = e^{i\lambda}$.

- Setting $\rho(u) = \sin(\lambda-u)/\sin\lambda$, the braid limits of the A-D-E face weights are given by the complex conjugate weights

$$\boxed{\begin{aligned} B\!\begin{pmatrix} d & c \\ a & b \end{pmatrix} &= \begin{array}{ccc} d & & c \\ & \boxed{-i\infty} & \\ a & & b \end{array} = \lim_{u\to -i\infty} \frac{x^{-\frac12}}{i\rho(u)}\, W\!\begin{pmatrix} d & c \\ a & b \end{pmatrix}\!\!\Big|\, u\Big) = -i\Big(x^{-\frac12}\,\delta_{ac} - x^{\frac12}\sqrt{\frac{\psi_a\psi_c}{\psi_b\psi_d}}\,\delta_{bd}\Big) \\ \overline{B}\!\begin{pmatrix} d & c \\ a & b \end{pmatrix} &= \begin{array}{ccc} d & & c \\ & \boxed{i\infty} & \\ a & & b \end{array} = \lim_{u\to i\infty} \frac{i\,x^{\frac12}}{\rho(u)}\, W\!\begin{pmatrix} d & c \\ a & b \end{pmatrix}\!\!\Big|\, u\Big) = i\Big(x^{\frac12}\,\delta_{ac} - x^{-\frac12}\sqrt{\frac{\psi_a\psi_c}{\psi_b\psi_d}}\,\delta_{bd}\Big) \end{aligned}}$$

Commuting Column Transfer Matrices

$[\mathbf{T}(u), \mathbf{T}(v)] = 0$



nimrep adjacency

$N \times M = 10\times 6$ lattice (torus)

seam transfer matrix

- A periodic lattice showing (i) the column/seam transfer matrices $\mathbf{T}(u)$, $\mathbf{B}$, $\widehat{\mathbf{N}}_a$, $\overline{\widehat{\mathbf{N}}}_b$ and (ii) the automorphism seam $\boldsymbol{\sigma}^\kappa$ with $\kappa = 0, 1$. The composite seam is the matrix product $\mathbf{T}_x = \boldsymbol{\sigma}^\kappa \widehat{\mathbf{N}}_a \overline{\widehat{\mathbf{N}}}_b$.

- The braid seams $\mathbf{B}, \overline{\mathbf{B}}$ are the limits $\lim_{u\to\mp i\infty} \mathbf{T}(u)$, that is, the face weights are replaced by the complex braid face weights $B, \overline{B}$. The seams $\mathbf{B}, \overline{\mathbf{B}}$ commute with $\mathbf{T}(u)$ and with each other. The fused s-type braid seams $\mathbf{n}_s$ (and their complex conjugates $\bar{\mathbf{n}}_s$) are defined recursively by

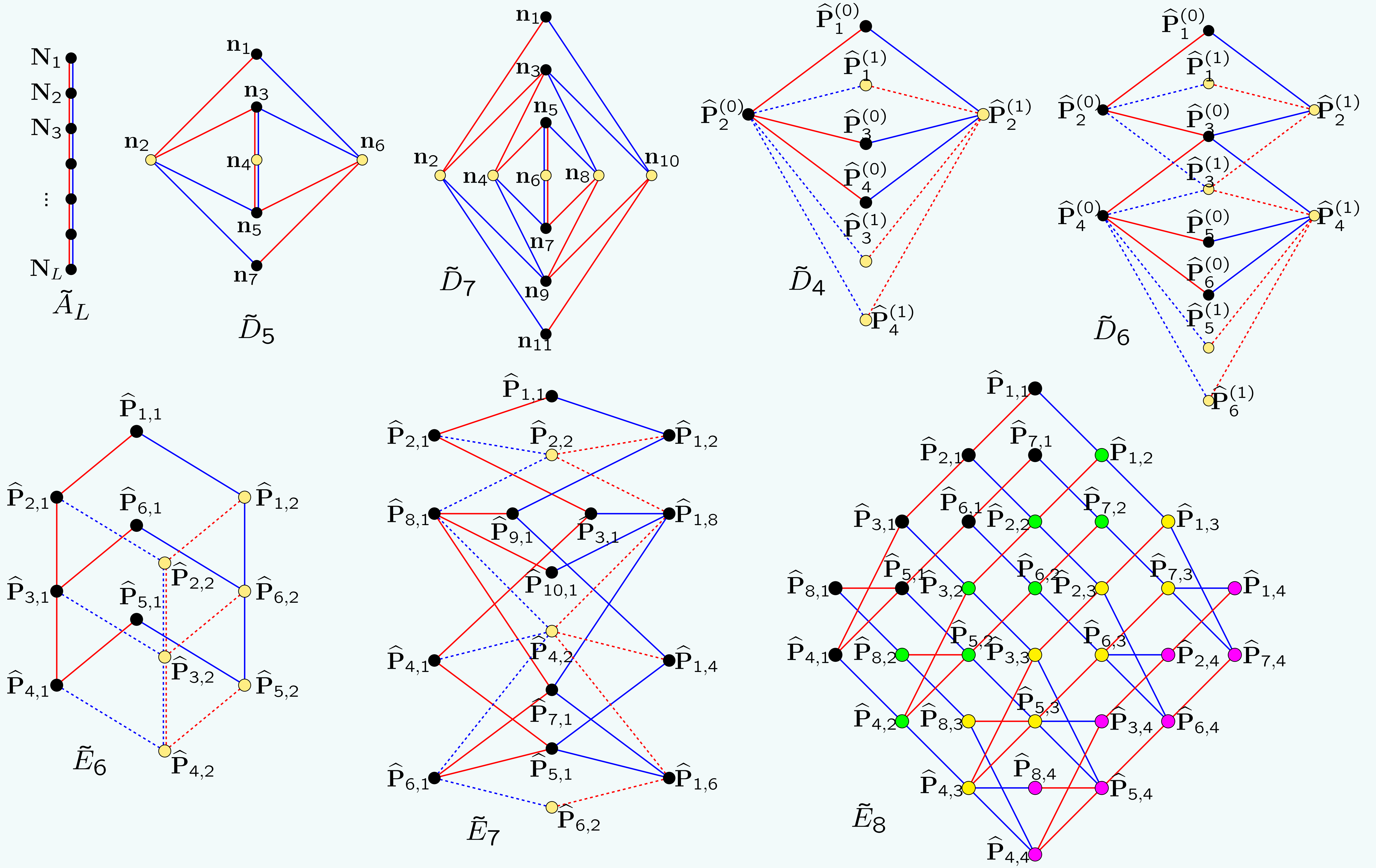
$$\mathbf{n}_1 = I, \qquad \mathbf{n}_2 = \mathbf{B}, \qquad \mathbf{n}_s = \mathbf{n}_2\mathbf{n}_{s-1} - \mathbf{n}_{s-2}, \qquad 3 \le s \le g-1; \qquad \mathbf{n}_j = U_{j-1}(\tfrac{1}{2}\mathbf{n}_2)$$

The $\widehat{\mathbf{N}}_a$ seams are given by the same linear combinations of $\mathbf{n}_s$ as the graph fusion matrices.

- For each A-D-E lattice model, the seams $\mathbf{n}_s, \bar{\mathbf{n}}_s$ and $\widehat{\mathbf{N}}_a, \overline{\widehat{\mathbf{N}}}_b$ satisfy the Verlinde and graph fusion algebras respectively for arbitrary system sizes:

$$\mathbf{n}_i\,\mathbf{n}_j = \sum_{k\in A_{g-1}} N_{ij}{}^k\,\mathbf{n}_k, \quad 1 \le i,j \le g-1; \qquad \widehat{\mathbf{N}}_a \widehat{\mathbf{N}}_b = \sum_{c\in G} \hat{N}_{ab}{}^c\,\widehat{\mathbf{N}}_c, \quad a,b \in G$$

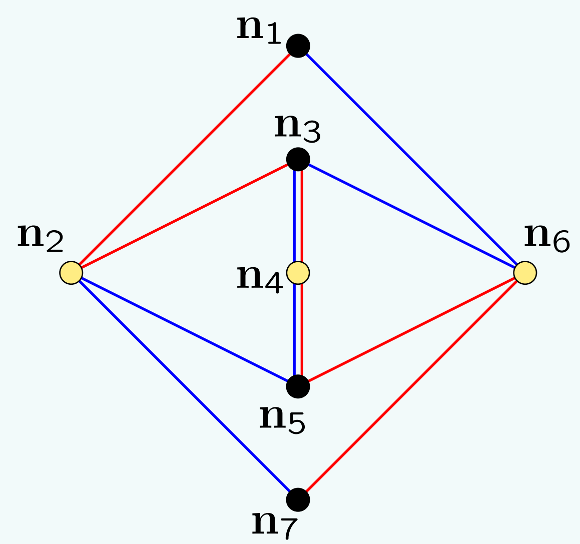
Ocneanu Fusion Graphs $\tilde{G}$



- The red/blue lines = action of the left/right chiral fundamentals $\widehat{\mathbf{N}}_2, \overline{\widehat{\mathbf{N}}_2}$ labelled by $a = 2, \bar{2}$.

$\tilde{D}_5$ Ocneanu Algebra of Seams

$\tilde{D}_5$



	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	$1+3$	$2+4$	$3+5$	$4+6$	$5+7$	6
3	3	$2+4$	$1+3+5$	$2+4+6$	$3+5+7$	$4+6$	5
4	4	$3+5$	$2+4+6$	$1+3+5+7$	$2+4+6$	$3+5$	4
5	5	$4+6$	$3+5+7$	$2+4+6$	$1+3+5$	$2+4$	3
6	6	$5+7$	$4+6$	$3+5$	$2+4$	$1+3$	2
7	7	6	5	4	3	2	1

- The Ocneanu fusion graph of $G = D_5$ with $H = A_7$ has 2 copies of the A_7 graph and 7 vertices. The fusion algebra coincides with the Ocneanu algebra of A_7. The Ocneanu graph fusion matrices of D_5 are the Verlinde matrices $\tilde{N}_j = N_j$, $j = 1, 2, \ldots, 7$:

$$\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix}
\begin{pmatrix}0&1&0&0&0&0&0\\1&0&1&0&0&0&0\\0&1&0&1&0&0&0\\0&0&1&0&1&0&0\\0&0&0&1&0&1&0\\0&0&0&0&1&0&1\\0&0&0&0&0&1&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0\\0&1&0&1&0&0&0\\1&0&1&0&1&0&0\\0&1&0&1&0&1&0\\0&0&1&0&1&0&1\\0&0&0&1&0&1&0\\0&0&0&0&1&0&0\end{pmatrix}
\begin{pmatrix}0&0&0&1&0&0&0\\0&0&1&0&1&0&0\\0&1&0&1&0&1&0\\1&0&1&0&1&0&1\\0&1&0&1&0&1&0\\0&0&1&0&1&0&0\\0&0&0&1&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&0&0&1&0&0\\0&0&0&1&0&1&0\\0&0&1&0&1&0&1\\0&1&0&1&0&1&0\\1&0&1&0&1&0&0\\0&1&0&1&0&0&0\\0&0&1&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&0&0&0&1&0\\0&0&0&0&1&0&1\\0&0&0&1&0&1&0\\0&0&1&0&1&0&0\\0&1&0&1&0&0&0\\1&0&1&0&0&0&0\\0&1&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&1&0\\0&0&0&0&1&0&0\\0&0&0&1&0&0&0\\0&0&1&0&0&0&0\\0&1&0&0&0&0&0\\1&0&0&0&0&0&0\end{pmatrix}$$

- Cayley table of the (commutative) D_5 Ocneanu seam algebra. Here a denotes the seam $\mathbf{n}_a$, $\overline{\mathbf{n}}_2 = \mathbf{n}_6$, $\overline{\mathbf{n}}_6 = \mathbf{n}_2$, $\boldsymbol{\sigma} = \mathbf{n}_7$ and $\overline{\mathbf{n}}_a = \mathbf{n}_a$ for the ambichiral nodes $a \in T_0 = \{1, 3, 4, 5, 7\}$:

$$\mathbf{n}_a\,\mathbf{n}_b = \sum_{c=1}^{7} \tilde{N}_{ab}{}^{c}\,\mathbf{n}_c, \qquad \tilde{N}_a \tilde{N}_b = \sum_{c=1}^{7} \tilde{N}_{ab}{}^{c}\,\tilde{N}_c, \qquad a, b = 1, 2, \ldots, 7$$

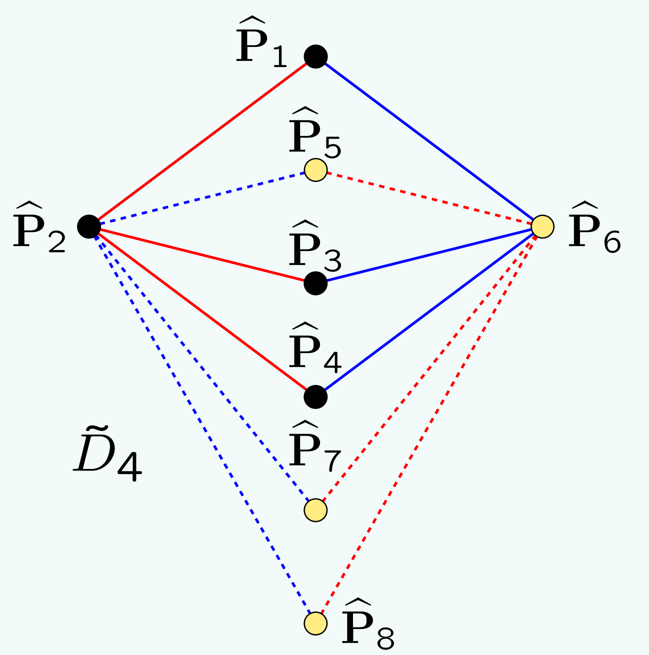
- Such seam relations are easily checked in Mathematica for system sizes $M \leq 12$.

$\tilde{D}_4$ Ocneanu Algebra of Seams

- The 8 Ocneanu graph fusion matrices (nimreps) $\tilde{N}_a$ for $\tilde{D}_4$ are

$$\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix}
\begin{pmatrix}0&1&0&0&0&0&0&0\\1&0&1&1&0&0&0&0\\0&1&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&1&0&1&1\\0&0&0&0&0&1&0&0\\0&0&0&0&0&1&0&0\end{pmatrix}
\begin{pmatrix}0&0&1&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&1&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0\end{pmatrix}
\begin{pmatrix}0&0&0&1&0&0&0&0\\0&1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&0&0&0&1&0&0\\0&0&0&0&1&0&1&1\\0&0&0&0&0&1&0&0\\0&0&0&0&0&1&0&0\\0&1&0&0&0&0&0&0\\1&0&1&1&0&0&0&0\\0&1&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&0&0&0&0&1&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0\\0&0&0&1&0&0&0&0\\0&1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\end{pmatrix}
\begin{pmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&1&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0\\0&0&1&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\\1&0&0&0&0&0&0&0\end{pmatrix}$$



	$\mathbf{I}$	$\mathbf{g}$	$\boldsymbol{\omega}$	$\boldsymbol{\omega}^2$	$\boldsymbol{\sigma}$	$\boldsymbol{\sigma}\mathbf{g}$	$\boldsymbol{\sigma\omega}$	$\boldsymbol{\sigma\omega}^2$
$\mathbf{I}$	$\mathbf{I}$	$\mathbf{g}$	$\boldsymbol{\omega}$	$\boldsymbol{\omega}^2$	$\boldsymbol{\sigma}$	$\boldsymbol{\sigma}\mathbf{g}$	$\boldsymbol{\sigma\omega}$	$\boldsymbol{\sigma\omega}^2$
$\mathbf{g}$	$\mathbf{g}$	$\mathbf{I}+\boldsymbol{\omega}+\boldsymbol{\omega}^2$	$\mathbf{g}$	$\mathbf{g}$	$\boldsymbol{\sigma}\mathbf{g}$	$\boldsymbol{\sigma}(\mathbf{I}+\boldsymbol{\omega}+\boldsymbol{\omega}^2)$	$\boldsymbol{\sigma}\mathbf{g}$	$\boldsymbol{\sigma}\mathbf{g}$
$\boldsymbol{\omega}$	$\boldsymbol{\omega}$	$\mathbf{g}$	$\boldsymbol{\omega}^2$	$\mathbf{I}$	$\boldsymbol{\sigma\omega}^2$	$\boldsymbol{\sigma}\mathbf{g}$	$\boldsymbol{\sigma}$	$\boldsymbol{\sigma\omega}$
$\boldsymbol{\omega}^2$	$\boldsymbol{\omega}^2$	$\mathbf{g}$	$\mathbf{I}$	$\boldsymbol{\omega}$	$\boldsymbol{\sigma\omega}$	$\boldsymbol{\sigma}\mathbf{g}$	$\boldsymbol{\sigma\omega}^2$	$\boldsymbol{\sigma}$
$\boldsymbol{\sigma}$	$\boldsymbol{\sigma}$	$\boldsymbol{\sigma}\mathbf{g}$	$\boldsymbol{\sigma\omega}$	$\boldsymbol{\sigma\omega}^2$	$\mathbf{I}$	$\mathbf{g}$	$\boldsymbol{\omega}$	$\boldsymbol{\omega}^2$
$\boldsymbol{\sigma}\mathbf{g}$	$\boldsymbol{\sigma}\mathbf{g}$	$\boldsymbol{\sigma}(\mathbf{I}+\boldsymbol{\omega}+\boldsymbol{\omega}^2)$	$\boldsymbol{\sigma}\mathbf{g}$	$\boldsymbol{\sigma}\mathbf{g}$	$\mathbf{g}$	$\mathbf{I}+\boldsymbol{\omega}+\boldsymbol{\omega}^2$	$\mathbf{g}$	$\mathbf{g}$
$\boldsymbol{\sigma\omega}$	$\boldsymbol{\sigma\omega}$	$\boldsymbol{\sigma}\mathbf{g}$	$\boldsymbol{\sigma\omega}^2$	$\boldsymbol{\sigma}$	$\boldsymbol{\omega}^2$	$\mathbf{g}$	$\mathbf{I}$	$\boldsymbol{\omega}$
$\boldsymbol{\sigma\omega}^2$	$\boldsymbol{\sigma\omega}^2$	$\boldsymbol{\sigma}\mathbf{g}$	$\boldsymbol{\sigma}$	$\boldsymbol{\sigma\omega}$	$\boldsymbol{\omega}$	$\mathbf{g}$	$\boldsymbol{\omega}^2$	$\mathbf{I}$

- Ocneanu fusion graph of D_4 with 4 copies of the D_4 graph. The 8 seams $\widehat{\mathbf{P}}_a^{(\kappa)} = \boldsymbol{\sigma}^\kappa \widehat{\mathbf{N}}_a$ are

$$\{\widehat{\mathbf{P}}_1, \widehat{\mathbf{P}}_2, \ldots, \widehat{\mathbf{P}}_8\} = \{\widehat{\mathbf{P}}_1^{(0)}, \widehat{\mathbf{P}}_2^{(0)}, \widehat{\mathbf{P}}_3^{(0)}, \widehat{\mathbf{P}}_4^{(0)}, \widehat{\mathbf{P}}_1^{(1)}, \widehat{\mathbf{P}}_2^{(1)}, \widehat{\mathbf{P}}_3^{(1)}, \widehat{\mathbf{P}}_4^{(1)}\} = \{\mathbf{I}, \mathbf{g}, \boldsymbol{\omega}, \boldsymbol{\omega}^2, \boldsymbol{\sigma}, \boldsymbol{\sigma}\mathbf{g}, \boldsymbol{\sigma\omega}, \boldsymbol{\sigma\omega}^2\}$$

with $\overline{\boldsymbol{\omega}} = \boldsymbol{\omega}^2$ and $\boldsymbol{\sigma}\mathbf{g} = \overline{\mathbf{g}} = \overline{\widehat{\mathbf{N}}}_2$. The extended-$\mathbb{S}_3$ $\tilde{D}_4$ Cayley table is noncommutative with $\boldsymbol{\sigma}^2 = \boldsymbol{\omega}^3 = \mathbf{I}$, $\boldsymbol{\omega\sigma} = \boldsymbol{\sigma\omega}^2$, $\boldsymbol{\omega}^2\boldsymbol{\sigma} = \boldsymbol{\sigma\omega}$. The seams $\boldsymbol{\sigma}, \boldsymbol{\omega}$ commute with the transfer matrix $\mathbf{T}(u)$.

- For system sizes $M \le 12$, the seams satisfy the Ocneanu algebra

$$\boxed{\widehat{\mathbf{P}}_a \widehat{\mathbf{P}}_b = \sum_{c=1}^{8} \tilde{N}_{ab}{}^c \widehat{\mathbf{P}}_c, \qquad \tilde{N}_a \tilde{N}_b = \sum_{c=1}^{8} \tilde{N}_{ab}{}^c \tilde{N}_c, \qquad a, b = 1, 2, \ldots, 8}$$

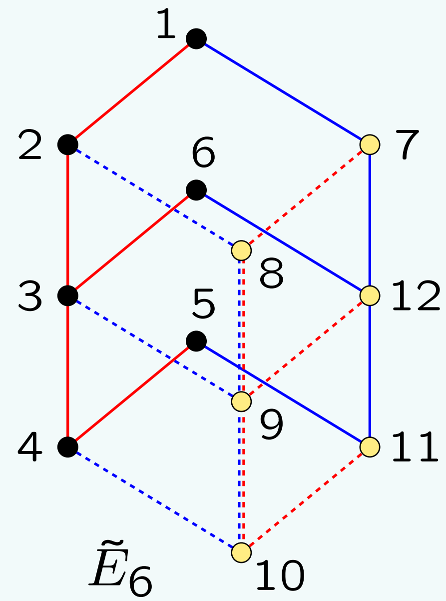
$\tilde{E}_6$ Ocneanu nimrpes

- The 12 Ocneanu graph fusion matrices (nimreps) $\tilde{N}_a$ for $\tilde{E}_6$ are

$$\begin{pmatrix}100000000000\\010000000000\\001000000000\\000100000000\\000010000000\\000001000000\\000000100000\\000000010000\\000000001000\\000000000100\\000000000010\\000000000001\end{pmatrix}
\begin{pmatrix}010000000000\\101000000000\\010101000000\\001010000000\\000100000000\\001000000000\\000000010000\\000000101000\\000000010101\\000000001010\\000000000100\\000000001000\end{pmatrix}
\begin{pmatrix}001000000000\\010101000000\\102010000000\\010101000000\\001000000000\\010100000000\\000000001000\\000000010101\\000000102010\\000000010101\\000000001000\\000000010100\end{pmatrix}
\begin{pmatrix}000100000000\\001010000000\\010101000000\\101000000000\\010000000000\\001000000000\\000000000100\\000000001010\\000000010101\\000000101000\\000000010000\\000000001000\end{pmatrix}
\begin{pmatrix}000010000000\\000100000000\\001000000000\\010000000000\\100000000000\\000001000000\\000000000010\\000000000100\\000000001000\\000000010000\\000000100000\\000000000001\end{pmatrix}
\begin{pmatrix}000001000000\\001000000000\\010100000000\\001000000000\\000001000000\\100010000000\\000000000001\\000000001000\\000000010100\\000000001000\\000000000001\\000000100010\end{pmatrix}$$

$$\begin{pmatrix}000000100000\\000000010000\\000000001000\\000000000100\\000000000010\\000000000001\\100000000001\\010000001000\\001000010100\\000100001000\\000010000001\\000001100010\end{pmatrix}
\begin{pmatrix}000000010000\\000000101000\\000000010101\\000000001010\\000000000100\\000000001000\\010000001000\\101000010101\\010101102010\\001010010101\\000100001000\\001000010100\end{pmatrix}
\begin{pmatrix}000000001000\\000000010101\\000000102010\\000000010101\\000000001000\\000000010100\\001000010100\\010101102010\\102010020202\\010101102010\\001000010100\\010100002000\end{pmatrix}
\begin{pmatrix}000000000100\\000000001010\\000000010101\\000000101000\\000000010000\\000000001000\\000100001000\\001010010101\\010101102010\\101000010101\\010000001000\\001000010100\end{pmatrix}
\begin{pmatrix}000000000010\\000000000100\\000000001000\\000000010000\\000000100000\\000000000001\\000010000001\\000100001000\\001000010100\\010000001000\\100000000001\\000001100010\end{pmatrix}
\begin{pmatrix}000000000001\\000000001000\\000000010100\\000000001000\\000000000001\\000000100010\\000001100010\\001000010100\\010100002000\\001000010100\\000001100010\\100010000002\end{pmatrix}$$

$$\Sigma = \begin{pmatrix}100000000000\\000000100000\\000000000001\\000000000010\\000010000000\\000001000000\\010000000000\\000000010000\\000000001000\\000000000100\\000100000000\\001000000000\end{pmatrix}$$



$$\boxed{\tilde{N}_a \, \tilde{N}_b = \sum_{c=1}^{12} \tilde{N}_{ab}{}^{c} \, \tilde{N}_c}$$

$$\tilde{N}_a = \begin{cases} \hat{N}_a \oplus \hat{N}_a, & a = 1, 2, \ldots, 6 \\ \tilde{N}_7 = \Sigma \, \tilde{N}_2 \Sigma, & \\ \tilde{N}_7 \tilde{N}_{a-6}, & a = 8, 9, \ldots, 12 \end{cases}$$

- $\Sigma = \mathbb{Z}_2$ chiral conjugation, $\Sigma \tilde{N}_a \Sigma = \tilde{N}_{\bar{a}}$, $\bar{2} = 7$, $\bar{3} = 12$, $\bar{4} = 11$ else $\bar{a} = a$ (ambichirals).

$\tilde{E}_6$ Ocneanu Algebra of Seams

- Following the relations for $\widehat{N}_a$:

$$\widehat{\mathbf{N}}_a = \mathbf{n}_a, \quad a = 1,2,3; \qquad \widehat{\mathbf{N}}_4 = \mathbf{n}_6 - \mathbf{n}_2, \quad \widehat{\mathbf{N}}_5 = \mathbf{n}_5 - \mathbf{n}_3 = \boldsymbol{\sigma}, \quad \widehat{\mathbf{N}}_6 = \mathbf{n}_2 + \mathbf{n}_4 - \mathbf{n}_6$$

with $\overline{\widehat{\mathbf{N}}}_b = \widehat{\mathbf{N}}_{\bar{b}}$, $\;b = 1,2,\ldots,6$. The 12 $\tilde{E}_6$ seams $\hat{\mathbf{P}}_{a,b} = \widehat{\mathbf{N}}_a \overline{\widehat{\mathbf{N}}}_b$ $(a = 1,2,\ldots,6;\ b = 1,2)$ are

$$\{\hat{\mathbf{P}}_1, \hat{\mathbf{P}}_2, \ldots, \hat{\mathbf{P}}_{12}\} = \{\hat{\mathbf{P}}_{1,1}, \hat{\mathbf{P}}_{2,1}, \ldots, \hat{\mathbf{P}}_{6,1}, \hat{\mathbf{P}}_{1,2}, \hat{\mathbf{P}}_{2,2}, \ldots, \hat{\mathbf{P}}_{6,2}\} \qquad \text{(Basis Seams)}$$

- The seams satisfy the Ocneanu algebra (checked in Mathematica for system sizes $M \le 12$)

$$\hat{\mathbf{P}}_\eta \hat{\mathbf{P}}_\mu = \sum_{\nu=1}^{12} \tilde{N}_{\eta\mu}{}^{\nu}\, \hat{\mathbf{P}}_\nu, \qquad \tilde{N}_\eta \tilde{N}_\mu = \sum_{\nu=1}^{12} \tilde{N}_{\eta\mu}{}^{\nu} \tilde{N}_\nu, \qquad \eta,\mu = 1,2,\ldots,12$$

- Cayley table of the commutative $\tilde{E}_6$ Ocneanu algebra using the notation $\eta = \hat{\mathbf{P}}_\eta$:

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	1+3	2+4+6	3+5	4	3	8	7+9	8+10+12	9+11	10	9
3	3	2+4+6	1+2(3)+5	2+4+6	3	2+4	9	8+10+12	7+2(9)+11	8+10+12	9	8+10
4	4	3+5	2+4+6	1+3	2	3	10	9+11	8+10+12	7+9	8	9
5	5	4	3	2	1	6	11	10	9	8	7	12
6	6	3	2+4	3	6	1+5	12	9	8+10	9	12	7+11
7	7	8	9	10	11	12	1+12	2+9	3+8+10	4+9	5+12	6+7+11
8	8	7+9	8+10+12	9+11	10	9	2+9	1+3+8 +10+12	2+4+6 +7+2(9)+11	3+5+8 +10+12	4+9	3+8+10
9	9	8+10+12	7+2(9)+11	8+10+12	9	8+10	3+8+10	2+4+6 +7+2(9)+11	1+2(3)+5 +2(8)+2(10)+2(12)	2+4+6 +7+2(9)+11	3+8+10	2+4+2(9)
10	10	9+11	8+10+12	7+9	8	9	4+9	3+5+8 +10+12	2+4+6 +7+2(9)+11	1+3+8 +10+12	2+9	3+8+10
11	11	10	9	8	7	12	5+12	4+9	3+8+10	2+9	1+12	6+7+11
12	12	9	8+10	9	12	7+11	6+7+11	3+8+10	2+4+2(9)	3+8+10	6+7+11	1+5+2(12)

- For $1 \le \eta \le 6$ and $\mu \ge 3$, $\hat{\mathbf{P}}_{\eta,\mu}$ is given in terms of the Basis Seams by the Quantum Symmetry

$$\hat{\mathbf{P}}_{\eta,\mu} = \widehat{\mathbf{N}}_\eta \overline{\widehat{\mathbf{N}}}_\mu = \widehat{\mathbf{N}}_\eta \widehat{\mathbf{N}}_{\bar{\mu}} = \sum_{\nu=1}^{12} \tilde{N}_{\eta\bar{\mu}}{}^{\nu}\, \hat{\mathbf{P}}_\nu, \qquad \eta,\mu = 1,2,\ldots,12$$

$\tilde{E}_6$ Ocneanu Algebra as a Quotient

- There is a ring homomorphism

$$\boxed{\phi : \left\langle \widehat{\mathbf{P}}_{a,1}, \widehat{\mathbf{P}}_{a,2} \,|\, a = 1, \ldots, 6 \right\rangle \to \mathbb{Z}[x,y] \big/ \left\langle p_1(x), p_2(x,y) \right\rangle}$$

$$p_1(x) = \prod_{s \in \mathrm{Exp}(E_6)} \left(x - 2\cos\frac{s\pi}{g} \right) = x^6 - 5x^4 + 5x^2 - 1, \qquad \mathrm{Exp}(E_6) = \{1,4,5,7,8,11\}$$

$$p_2(x,y) = y^2 + (x^5 - 5x^3 + 4x)y - 1, \qquad p_2(y,y) = p_1(y)$$

$$\widehat{\mathbf{P}}_{1,1} \mapsto 1, \quad \widehat{\mathbf{P}}_{2,1} \mapsto x, \quad \widehat{\mathbf{P}}_{3,1} \mapsto x^2 - 1, \quad \widehat{\mathbf{P}}_{4,1} \mapsto x^5 - 4x^3 + 2x$$

$$\widehat{\mathbf{P}}_{5,1} \mapsto x^4 - 4x^2 + 2, \quad \widehat{\mathbf{P}}_{6,1} \mapsto -x^5 + 5x^3 - 4x, \quad \widehat{\mathbf{P}}_{a,2} \mapsto \phi(\widehat{\mathbf{P}}_{a,1})\, y, \quad a = 1, 2, \ldots, 6$$

- The $\tilde{E}_6$ integrable seams mutually commute and are simultaneously diagonalizable.

The eigenvalues yield a 1-d representation of the $\tilde{E}_6$ Ocneanu algebra in terms of the quantum dimensions. Set $x = q^s + q^{-s} = 2\cos\frac{s\pi}{12}$, $y = q^{\bar{s}} + q^{-\bar{s}} = 2\cos\frac{\bar{s}\pi}{12}$ with $q = e^{\pi i/12}$

- Solving

$$\boxed{p_1(x) = 0, \qquad p_2(x,y) = 0}$$

yields $6 \times 2 = 12$ solutions for x, y:

$$\boxed{(s,\bar{s}) \in \mathcal{S} = \{(1,1),(4,4),(5,5),(7,7),(8,8),(11,11),(1,7),(7,1),(4,8),(8,4),(5,11),(11,5)\}}$$

These indices coincide with the operator content of the E_6 modular invariant partition function

$$\boxed{Z(q) = \sum_{r=1}^{g-2} \sum_{(s,\bar{s}) \in \mathcal{S}} \chi_{r,s}(q) \chi_{r,\bar{s}}(\bar{q})}$$

E_6 Quantum Dimensions

- As required, the set $\mathcal{S}$ is symmetric under interchange of s and $\bar{s}$. In the $(s,\bar{s})$ sector, the quantum dimensions giving the 1-dimensional representation of $\tilde{E}_6$ are

$(1,1)\colon 1$

$(2,1)\colon q^s+q^{-s}$

$(3,1)\colon q^{2s}+1+q^{-2s}$

$(4,1)\colon q^{5s}+q^{3s}+q^{-3s}+q^{-5s}$

$(5,1)\colon q^{4s}+q^{-4s}$

$(6,1)\colon -q^{5s}+q^s+q^{-s}-q^{-5s}$

$(1,2)\colon q^{\bar{s}}+q^{-\bar{s}}$

$(2,2)\colon q^{s+\bar{s}}+q^{s-\bar{s}}+q^{-s+\bar{s}}+q^{-s-\bar{s}}$

$(3,2)\colon q^{2s+\bar{s}}+q^{2s-\bar{s}}+q^{\bar{s}}+q^{-\bar{s}}+q^{-2s+\bar{s}}+q^{-2s-\bar{s}}$

$(4,2)\colon q^{5s+\bar{s}}+q^{5s-\bar{s}}+q^{3s+\bar{s}}+q^{3s-\bar{s}}+q^{-3s+\bar{s}}+q^{-3s-\bar{s}}+q^{-5s+\bar{s}}+q^{-5s-\bar{s}}$

$(5,2)\colon q^{4s+\bar{s}}+q^{4s-\bar{s}}+q^{-4s+\bar{s}}+q^{-4s-\bar{s}}$

$(6,2)\colon -q^{5s+\bar{s}}-q^{5s-\bar{s}}+q^{s+\bar{s}}+q^{s-\bar{s}}+q^{-s+\bar{s}}+q^{-s-\bar{s}}-q^{-5s+\bar{s}}-q^{-5s-\bar{s}}$

- Taking the union within each of the 12 $(s,\bar{s})$ sectors, the distinct eigenvalues of the integrable seams are given by the roots of the minimal polynomials:

$(1,1)$:	$\lambda-1$	$(1,2)$:	$p_1(\lambda)$
$(2,1)$:	$p_1(\lambda)$	$(2,2)$:	$\lambda^4-4\lambda^3+4\lambda-1$
$(3,1)$:	$\lambda^3-2\lambda^2-2\lambda$	$(3,2)$:	$\lambda^7-30\lambda^5+60\lambda^3-8\lambda$
$(4,1)$:	$p_1(\lambda)$	$(4,2)$:	$\lambda^4-4\lambda^3+4\lambda-1$
$(5,1)$:	λ^2-1	$(5,2)$:	$p_1(\lambda)$
$(6,1)$:	$\lambda^3-2\lambda$	$(6,2)$:	$\lambda^3-2\lambda^2-2\lambda$

- Some integrable seams (red) have a zero eigenvalue and so are not invertible. Strictly speaking, in this sense, these seams are not symmetries — they are non-invertible symmetries.

$\tilde{E}_6$ Structure Constants

$$\text{For } \tilde{E}_6: \qquad p_2(\mathbf{B},\overline{\mathbf{B}}) = \overline{\mathbf{B}}^2 - \widehat{\mathbf{P}}_{6,1}\overline{\mathbf{B}} - \mathbf{I} = \mathbf{0} \quad \Leftrightarrow \quad \overline{\widehat{\mathbf{N}}}_2^{\,2} = \mathbf{I} + \widehat{\mathbf{N}}_6\overline{\widehat{\mathbf{N}}}_2$$

- Using this, the product of any two seams $\widehat{\mathbf{P}}_{a,a'}$ and $\widehat{\mathbf{P}}_{b,b'}$ is obtained using commutativity and associativity. Let $a,b,c = 1,2,\ldots,6$ and $a',b',c' = 1,2$ and consider the four quadrants (a',b') of the Cayley table. By commutativity the $(1,2)$ and $(2,1)$ quadrants agree leaving 3 cases:

$$a'b' = 1: \quad \widehat{\mathbf{N}}_a\widehat{\mathbf{N}}_b = \sum_c \widehat{N}_{ab}{}^c\,\widehat{\mathbf{N}}_c = \sum_c \widehat{N}_{ab}{}^c\,\widehat{\mathbf{P}}_{c,1}$$

$$a'b' = 2: \quad (\widehat{\mathbf{N}}_a\overline{\widehat{\mathbf{N}}}_2)\widehat{\mathbf{N}}_b = \widehat{\mathbf{N}}_a(\widehat{\mathbf{N}}_b\overline{\widehat{\mathbf{N}}}_2) = \sum_c \widehat{N}_{ab}{}^c(\widehat{\mathbf{N}}_c\overline{\widehat{\mathbf{N}}}_2) = \sum_c \widehat{N}_{ab}{}^c\,\widehat{\mathbf{P}}_{c,2}$$

$$\begin{aligned} a'b' = 4: \quad (\widehat{\mathbf{N}}_a\overline{\widehat{\mathbf{N}}}_2)(\widehat{\mathbf{N}}_b\overline{\widehat{\mathbf{N}}}_2) &= \sum_c \widehat{N}_{ab}{}^c\,\widehat{\mathbf{N}}_c\,\overline{\widehat{\mathbf{N}}}_2^{\,2} = \sum_c \widehat{N}_{ab}{}^c\,\widehat{\mathbf{N}}_c\,(\mathbf{I} + \widehat{\mathbf{N}}_6\overline{\widehat{\mathbf{N}}}_2) \\ &= \sum_c \widehat{N}_{ab}{}^c\,\widehat{\mathbf{N}}_c + \sum_{c,d} \widehat{N}_{ab}{}^c\,\widehat{N}_{c6}{}^d(\widehat{\mathbf{N}}_d\overline{\widehat{\mathbf{N}}}_2) = \sum_c \widehat{N}_{ab}{}^c\,\widehat{\mathbf{P}}_{c,1} + \sum_{c,d} \widehat{N}_{ab}{}^c\,\widehat{N}_{c6}{}^d\widehat{\mathbf{P}}_{d,2} \end{aligned}$$

This gives the $\tilde{E}_6$ Ocneanu algebra structure constants in terms of $\widehat{N}_{ab}{}^c$ as

$$\widetilde{N}_{a,a';\,b,b'}{}^{c,c'} = (\delta_{a',b'}\delta_{c',1} + \delta_{a'b',2}\delta_{c',2})\widehat{N}_{ab}{}^c + \delta_{a'b',4}\delta_{c',2}\sum_{d=1}^{6}\widehat{N}_{ab}{}^d\widehat{N}_{d6}{}^c = \text{twelve } 12\times 12 \text{ matrices}$$

- Let us prove $p_2(\mathbf{B},\overline{\mathbf{B}}) = \mathbf{0}$. For a given u-independent eigenvector of $\mathbf{T}(u)$, the eigenvalues of $\mathbf{B},\overline{\mathbf{B}}$ are $2\cos\frac{s\pi}{12}$, $2\cos\frac{\bar{s}\pi}{12}$ with $(s,\bar{s}) \in \mathcal{S}$. But it is readily verified that

$$p_2(2\cos\tfrac{s\pi}{12}, 2\cos\tfrac{\bar{s}\pi}{12}) = 0, \qquad (s,\bar{s}) \in \mathcal{S}$$

The proof that the integrable seams satisfy the $\tilde{E}_6$ Ocneanu algebra, for arbitrary system sizes M, thus follows by simultaneous diagonalization since the seam eigenvalues (quantum dimensions) satisfy the $\tilde{E}_6$ Ocneanu algebra.

A-D-E Minimal CFTs

- The conformal symmetry algebra is $\text{Vir}\otimes\overline{\text{Vir}}$ on the torus. A rational CFT [MS89] admits a finite number of irreducible representations of this algebra which close under fusion. An irrep is characterized by associated conformal weights $(\Delta,\overline{\Delta})$. A CFT is unitary if the conformal weights are all positive.

- The unitary rational CFTs with central charges $c<1$ and their modular invariant conformal partition functions are classified [CIZ87] by the simply-laced A-D-E Lie algebras. For $g=4,5,\ldots$, these minimal CFTs [BPZ84] have central charges and conformal weights

$$\boxed{c=1-\frac{6}{g(g-1)}=\frac{1}{2},\frac{7}{10},\frac{4}{5},\ldots\qquad \Delta_{r,s}=\frac{(gr-(g-1)s)^2-1}{4g(g-1)},\quad r=1,2,\ldots,g-2;\ \ s=1,2,\ldots,g-1}$$

- The continuum scaling limit of the critical A-D-E RSOS lattice models yields precisely the A-D-E minimal CFTs with the 1-1 correspondences:

$$\boxed{\text{Vir irrep}}\ \leftrightarrow\ \boxed{\text{conformal defect}}\ \leftrightarrow\ \boxed{\text{integrable seam}}$$

The integrable seams are constructed using fusions which label the conformal defects.

- The fusions are encoded in the graph fusion algebras and their nimreps. These structures are already present for finite lattices! The nimreps give the adjacency conditions across the elementary integrable seam transfer matrices.

- The conformal defects/seams are topological in the sense that they propagate along freely and pass through each other.

A-D-E Twisted Conformal Partition Functions

- The A-D-E CFTs are cosets of the form $(A_{g-2}, G) = (A_{g-2} \otimes G)/\mathbb{Z}_2$. The conformal partition functions of the irreps of the A-D-E minimal CFTs are sesquilinear forms in Virasoro characters $\chi_{r,s}(q)$ [PZ2001] on the torus:

$$\boxed{Z_{\text{torus}}^{(r,a,b,\kappa)} = \begin{cases} \displaystyle\sum_{(r',s'),(r'',s'')} N_{r,r'}{}^{r''} [P_{a,b}^{(\kappa)}]_{s's''}\, \chi_{r',s'}(q)\, \chi_{r'',s''}(\bar{q}), & \text{Types I and II} \\ \displaystyle\sum_{(r',a'),(r'',a'')} N_{r,r'}{}^{r''} [\widehat{P}_{a,b}^{(\kappa)}]_{a'a''}\, \widehat{\chi}_{r',a'}(q)\, \widehat{\chi}_{r'',a''}(\bar{q}), & \text{Type I} \end{cases}}$$

The second form is new. Here $r, r', r'' \in A_{g-2}$; $s, s', s'' \in A_{g-1}$; $a, a', a'', b \in G$; $\kappa = 0, 1$. The label $\kappa = 0, 1$ is only needed for the D_L series with L even. We do not discuss r-type seams here.

- The *toric matrices*, *extended toric matrices* and extended characters are

$$\boxed{[P_{a,b}^{(\kappa)}]_{s's''} = \sum_{c \in T_\kappa} n_{s'a}{}^{c}\, n_{s''b}{}^{\zeta(c)}, \qquad [\widehat{P}_{a,b}^{(\kappa)}]_{a'a''} = \sum_{c \in T_\kappa} \widehat{N}_{ac}{}^{a'}\, \widehat{N}_{bc}{}^{a''}, \qquad \widehat{\chi}_{r,a}(q) = \sum_{s=1}^{g-1} n_{s1}{}^{a} \chi_{r,s}(q)}$$

The sets T_0 and T_1 are

$$\boxed{T_0 = \begin{cases} \{1, 2, \ldots, L\}, & G = A_L \\ \{1, 3, 5, \ldots, 2\ell-1, 2\ell\}, & G = D_{2\ell} \\ \{1, 5, 6\}, & G = E_6 \\ \{1, 7\}, & G = E_8 \end{cases} \qquad T_1 = \begin{cases} \{2, 4, \ldots, 2\ell-2\}, & G = D_{2\ell} \\ T_0, & \text{otherwise} \end{cases}}$$

The involutive twist is $\zeta(c) = c$ is trivial except for the type II cases

$$D_{2\ell+1}: \quad \zeta(c) = 4\ell - c, \quad c \in A_{4\ell-1} \text{ even}; \qquad E_7 : H = D_{10} : \quad \zeta(3) = 9, \quad \zeta(9) = 3$$

Conclusion

- It is argued that, for all critical (unitary) A-D-E lattice models and arbitrary system sizes M:

(i) the s-type braid transfer matrices $\mathbf{n}_s$ satisfy the Verlinde algebra,
(ii) the a-type integrable seams $\widehat{\mathbf{N}}_a$ satisfy the graph fusion algebra,
(iii) the integrable seams $\widehat{\mathbf{P}}_{a,b}$ satisfy the Ocneanu algebra and quantum symmetry.
All of these statements have been checked in Mathematica for $M \le 12$.

- Expressions can be obtained for the local face weights of each of the integrable seams. For Type I theories, explicit expressions are obtained for the Ocneanu algebra structure constants.

- Conformal topological defects result from the integrable seams in the continuum scaling limit.

- The algebra of seams gives a natural physical interpretation of the Ocneanu algebra — it implements the local operator product expansion on conformal defects. The Ocneanu algebra encodes how, for twisted boundaries, the left and right chiral halves of a CFT are glued together.

- Since the seams commute with the transfer matrix $\mathbf{T}(u)$, they are in some sense symmetries. But some seams admit zero eigenvalues, so these seams are generalized noninvertible symmetries.

- The A-D-E models are $s\ell(2)$ spin-$\frac{1}{2}$ models. The new insights give tools to systematically investigate the Ocneanu algebras of higher spin (eg. spin-1) and higher rank models (eg. $s\ell(3)$ models) as well as $s\ell(2)$ affine A-D-E models and logarithmic minimal models.

- Related results have also been obtained for (r,a) type defects of unitary and nonunitary A-D-E RSOS models on the cylinder (Pearce,Heymann,Quella; in preparation).