Integrable structure emerges from gauge theories

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Motivation

In this talk, I will tell you that the spin chain can emerge from a two-dimensional supersymmetric gauge theory.
In memory of Baxter and Yang: 2D SUSY Yang–Mills theory can be mapped to a 2D quantum integrable system (the Yang–Baxter equation).
But why is this interesting?

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- Physicists have found several similarities between supersymmetric gauge theory and integrable systems. For example, the vacuum equations of the gauge theory coincide with the Bethe–ansatz equations of the corresponding integrable system.
- Possible others..

One of my goals in this talk is to convince you that the third point is a fact, then it explains why the three "different subjects" are inherently related.

Integrability in the 2D compact BF Theory

Let us start from a simple situation

- Fields: U(1) gauge field A and compact scalar $\phi \sim \phi + 2\pi$.
- Action:

$$S = \frac{k}{2\pi} \int_{\Sigma} \phi \, F, \qquad F = dA.$$

- Level $k \in \mathbb{Z}$ ensures gauge invariance (compactness of A and ϕ).
- No local degrees of freedom: the theory is topological.

Classical Structure & Constraints

- Equations of motion:
$$F = dA = 0, \qquad d\phi = 0.$$
- A is flat $\Rightarrow$ only holonomy around cycles survives.
- ϕ is constant on each connected component of the spatial slice.
- All degrees of freedom are global/topological.

BF theory on $\mathbb{R} \times S^1$

Consider spacetime $M = \mathbb{R}_t \times S^1_x$.

- Integrate out A_0 gives that $\phi = \phi(t)$
- define the spatial zero modes:

$$a(t) := \oint_{S^1} A_1, \qquad \phi_0(t).$$

- then we have

$$L_{\text{eff}} = \frac{k}{2\pi}\,\phi_0(t)\,\dot{a}(t).$$

- First-order system $\Rightarrow$ symplectic structure is directly visible from L_{eff}.

Action–angle coordinates

See Nekrasov Witten 2010 for more

- Momentum conjugate to a:
$$p_a = \frac{\partial L_{\text{eff}}}{\partial \dot{a}} = \frac{k}{2\pi}\,\phi_0.$$
- Define action–angle variables:
$$\theta := a, \qquad I := p_a.$$
- Canonical one-form:
$$\Theta = I\,d\theta.$$
- Symplectic form:
$$\omega = d\Theta = dI \wedge d\theta.$$

Phase Space & Quantization

- Periodicities:
$$\theta \sim \theta + 2\pi, \qquad \phi_0 \sim \phi_0 + 2\pi \Rightarrow I = \frac{k}{2\pi}\phi_0 \sim I + k.$$
- Phase space:
$$\mathcal{P} \simeq S^1_\theta \times S^1_I \quad \text{(a 2-torus)}.$$
- Symplectic area:
$$\int_{\mathcal{P}} \omega = 2\pi k.$$
- Quantization $\Rightarrow$ finite-dimensional Hilbert space,
$$\dim \mathcal{H} = k.$$
- Pure BF: $H = 0 \Rightarrow$ frozen dynamics (no time evolution).

Summary

- 2D compact BF theory is exactly solvable and topological.
- On S^1 it reduces to a single effective mechanical degree of freedom.
- Natural action–angle description (θ, I) with torus phase space. It has the holomorphic/algebraic integrable systems
- **To get the usual quantum integrable system, the above is certainly not enough, we need more..**

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Many of you may be already familiar with the spin chain. It has rich important physics, but the setup is simple.
It has N-sites, each site has a vector space $\mathcal{V}$ and the associated Hamiltonian as

$$H = \sum_{i=1}^{N} \left(S_{+,i} S_{-,i-1} + S_{-,i} S_{+,i-1}\right) + \ldots$$

Figure: Spin chain.

Spin chain

Examples. Consider a spin-1/2 system: $\mathcal{V} = \{+, -\}$.
• XX model, $H = \sum_{i=1}^{N} \left(S_{+,i} S_{-,i-1} + S_{-,i} S_{+,i-1} \right)$.

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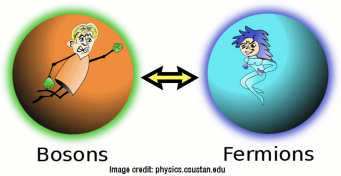
$\cdots$
The k spin-up configuration corresponds to the configuration of k-magnons.
We will see that these can be mapped to the supersymmetric gauge theory. Magnons map to the vacuum states of supersymmetric gauge theory, and the interaction $S_{+,i}S_{-,i-1}$ and $S_{-,i}S_{+,i-1}$ map to the fundamental domain walls $\cdots$

We need supersymmetry



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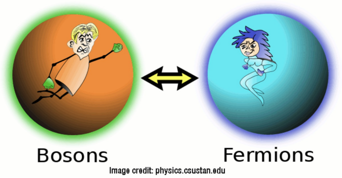

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I only focus on the theoretical aspects of SUSY:

- The ground states of supersymmetric field theory all have the vanishing energy, and it is renormalization group (RG) flow invariance.
- BPS spectrum is also RG-protected.

A brief review of 2d $\mathcal{N}$=(2,2) GLSMs

Some relevant (and new) points:

- Setup.
  - I only focus on $U(k)$ gauge group in this talk. Super gauge field V, its field strength Σ with F_{01} to be one component field.
  - A twisted superpotential $\tilde{W} = -t \cdot tr\Sigma$, $t = r - i\theta$, where r is FI-parameter, θ is the usual theta angle in a gauge theory.

$$\Sigma = \sigma + i\theta^+\bar{\lambda}_+ - i\bar{\theta}^-\lambda_- + \dots, \quad \bar{\Sigma} = \bar{\sigma} - i\bar{\theta}^+\lambda_+ + i\theta^-\bar{\lambda}_- + \dots$$

  - This talk only considers fundamental matter fields Φ_i. For N matter fields, the global symmetry is $SU(N)$. One can break this global symmetry by turning on the so-called the twisted masses $m_i = \langle\Sigma'\rangle$, where Σ' is the superfield strength of the background gauge field V' for $SU(N)$ global symmetry.

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- Phases $\simeq$ ground states in a semi-classical region. Symmetries are important.
  • Lorentz symmetry: $U(t; \phi_i, F_{01})$. Perturbative RG-flow $r = \sum_i \rho_i \log \frac{\mu}{\Lambda}$.
  • SUSY: $U = 0$.
  • One-form symmetries: F_{01} was missed in the previous studies. It is important! See my work '21 on the emergent dynamical decomposition.
  • If the semi-classical analysis is not trustful, then we consider the effective theory by integrating out matters:

$$\widetilde{W}_{eff} = -(t + i(k-1)\pi)\sum_{a=1}^{k}\Sigma_a - \sum_{i,a}\rho_i^a\Sigma_a\left(\log\left(\rho_i^a\Sigma_a\right) - 1\right).$$

Embed the compact BF theory into the GLSM

Let us focus on the abelian situation first. The bosonic part of the the twisted effective action has the following term

$$NF_{01} \arg(\sigma/\Lambda)$$

If we map

$$\sigma\Lambda^{-1} \mapsto e^{i\phi},$$

then we get the Lagrangian of the compact BF model. One can also do this for a nonabelian group.

Gr(k;N)

Gauged linear sigma model for Gr(k;N) is well-studied. See Witten '93, Gu-Sharpe '18. It is a $U(k)$ gauge theory with N fundamental fields.

- When $r \gg 0$, the semi-classical vacuum configuration is the geometry: Gr(k;N).
- When $r \ll 0$, the vacuum structure is described by a twisted effective superpotential

$$\widetilde{W}_{eff} = -(t + i(k-1)\pi) \sum_{a=1}^{k} \Sigma_a - \sum_a N\Sigma_a \left(\log\left(\Sigma_a\right) - 1\right).$$

  The vacuum equations

$$e^{\frac{\partial \widetilde{W}_{eff}}{\partial \sigma_a}} = 1$$

  give

$$(\sigma_a)^N = \widetilde{q}, \quad \widetilde{q} = e^{-t-i\pi(k-1)}, \qquad \mathrm{a} \in \{1, \cdots, k\}, \quad \sigma_a \neq \sigma_b.$$

We will see the above equations are also the Bethe-ansatz equations of the N-sites XX-spin chain.

BPS spectrum

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$$Z \approx \Lambda$$

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It has a finite mass, so, unlike the bosonic theory, 2d $\mathcal{N} = (2, 2)$ massive theory is not confined. More generally, we have

$$Z_{\ell_1 \ell_2} = W\mid_{\ell_2} - W\mid_{\ell_1}, \qquad \ell \text{ labels the vacua.}$$

See Hannay-Hori '97 and Hori-Vafa '20 for abelian theories. See also Dorey '98 about the similarity to the BPS-spectrum in Seiberg-Witten theory. For nonabelian theories, see Gu '22. If turning on the twisted mass, the central charge expression will be modified.

An intermediate scale

In quantum field theory, the physics depends on scale. Now if we consider a scale

$$\Lambda \ll e(\mu),$$

where e is the gauge coupling. The mass of the perturbative spectrum is proportional to the gauge coupling e. At this scale, the physics degrees of freedom are domain walls. We will show that this is the theory for spin chains.

State space for vacua

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From the structure, we can enlarge the fermionic degrees of freedom as

$$\overline{\lambda}_{\pm,j} := \int d^2\sigma \delta^2(\sigma - \overline{\sigma}_j)\overline{\lambda}_{\pm}, \qquad \lambda_{\pm,j} := \int d^2\sigma \delta^2(\sigma - \overline{\sigma}_j)\lambda_{\pm},$$

and their canonical commutation relations

$$\{\overline{\lambda}_{\pm,j}, \lambda_{\pm,i}\} = \delta_{ji}.$$

State space

If we replace the formal state

$$|0\rangle \mapsto \prod_{i=1}^{N-k} \bar{\lambda}_{-,i}|\Omega\rangle$$

Now we can represent the state-space by a formal notation, for example, for k=2

$$|-,\cdots,+,-,\cdots,-,+,-,\cdots,-\rangle.$$

This notation is not necessary for quantum field theory, however, it would be useful for presenting the data of the spin chain.

Spin operators

One can construct local spin operators on each site as

$$S_{+,i} = \bar{\lambda}_{+,i}\lambda_{-,i}, \quad S_{-,i} = \bar{\lambda}_{-,i}\lambda_{+,i}, \quad S_{z,i} = \frac{1}{2}\left(\bar{\lambda}_{+,i}\lambda_{+,i} - \bar{\lambda}_{-,i}\lambda_{-,i}\right).$$

It is readily to show that

$$[S_+, S_-] = 2S_z.$$

This explains why we need $\mathcal{N} = (2,2)$ supersymmetry for getting a spin chain. For more general global symmetry in gauge theory, it would have one more index for defining the local spin operators, see Gu '22.

The correspondence of domain walls in spin chain

The domain wall can be regarded as a map such as

$$|-, \cdots, -_{i-1}, +_i, -_{i+1} \cdots\rangle \mapsto |-, \cdots, -_{i-2}, +_{i-1}, -_i \cdots\rangle$$

$$\phi :\mapsto \mathcal{D}_i = S_{-,i} S_{+,i-1}.$$

One can similarly define the anti-domain wall. For more general domain walls, a similar construction applies.

Constraints from domain walls

In our case, we have the global symmetry $SU(N)$ in our target space. The domain walls are charged under the center group $\mathbb{Z}_N$ of the flavor group $SU(N)$. So the dynamic domain walls suggested that all of the physical state space should be neutral under the center group $\mathbb{Z}_N$. Notice that, the expectation value of σ field is also charged under $\mathbb{Z}_N$ group (also they are charged under the axial-R symmetry as well), which can be used to construct the states.

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Notice that, the expectation value of σ field is also charged under $\mathbb{Z}_N$ group (also they are charged under the axial-R symmetry as well), which can be used to construct the states.
For example, consider the k=1 case, the physical state space at the intermediate scale can be represented as

$$\sum_{i=1}^{N} \sigma^i \mid \cdots +_i \cdots \rangle.$$

This is actually the state for one "magnon" in the spin chain if we replace σ with e^{ip}, where p is the momentum of the magnon.

Finite symmetries: CPT

CPT-symmetries in quantum field theory can also descend to the spin chain. They act as the following:

$$\begin{aligned}
\mathcal{P} S_{\pm,i} \mathcal{P}^{-1} &= S_{\mp,i} \qquad & \mathcal{P} S_{z,i} \mathcal{P}^{-1} &= -S_{z,i}, \\
\mathcal{T} S_{\pm,i} \mathcal{T}^{-1} &= -S_{\pm,i} \qquad & \mathcal{T} S_{z,i} \mathcal{T}^{-1} &= S_{z,i}, \\
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$$\begin{aligned}
\mathcal{P}\mathcal{D}_i\mathcal{P}^{-1} &= \bar{\mathcal{D}}_i & \qquad \mathcal{P}\mathcal{D}_N\mathcal{P}^{-1} &= \left(\frac{\widetilde{q}}{|\widetilde{q}|}\right)^2 \bar{\mathcal{D}}_N, \\
\mathcal{T}\mathcal{D}_i\mathcal{T}^{-1} &= \mathcal{D}_i & \qquad \mathcal{T}\mathcal{D}_N\mathcal{T}^{-1} &= \left(\frac{\widetilde{q}}{|\widetilde{q}|}\right)^{-2} \mathcal{D}_N, \\
\mathcal{C}\mathcal{D}_i\mathcal{C}^{-1} &= \bar{\mathcal{D}}_i & \qquad \mathcal{C}\mathcal{D}_N\mathcal{C}^{-1} &= \bar{\mathcal{D}}_N,
\end{aligned}$$

where we have used the fact that $\mathcal{T}i\mathcal{T}^{-1} = -i$ for $i^2+1=0$. From the above, one can observe that $\mathcal{P}$ and $\mathcal{T}$ may be violated individually unless $\widetilde{q} = \pm 1$. However, we will see that the scattering factor could also break the $\mathcal{T}$-symmetry if it is not a pure phase factor.

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• When the scattering factor is generic not a pure phase: It is always an open spin chain for any finite $\widetilde{q}$.

Hamiltonian

For a closed spin chain, it is natural to propose the Hamiltonian as

$$H = \sum_i \mathcal{D}_i + \sum_i \bar{\mathcal{D}}_i + f(\mathcal{D}_i, \bar{\mathcal{D}}_i),$$

where the factor $f(\mathcal{D}_i, \bar{\mathcal{D}}_i)$ can be fixed if we require that the state spaces are eigenstates of this Hamiltonian. We have assumed the fact that all other matter representations are a subspace of the tensor product of fundamental matters. This is not true for general, and for a general case, one may need to take into account the so-called higher symmetries. We will not focus on this in this talk.

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For an open spin chain, we take the (anti-)holomorphic part of the above.

A summary

Let us summarize what we have. Before, Nekrasov and Shatashvili found about the correspondence between the integrable system and the gauge theory:

$\Upsilon(p_a)$	$\leftrightarrow$	$\widetilde{W}_{\mathrm{eff}}(\sigma_a)$
p_a	$\leftrightarrow$	σ_a
k-particle sector	$\leftrightarrow$	gauge group $U(k)$
N-sites	$\leftrightarrow$	flavor group $SU(N)$
twisted boundary	$\leftrightarrow$	$t = r - i\theta$
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Based on the vacuum structure and the dynamics of the domain walls. We have shown that the spin chain can be constructed from the $\mathcal{N} = (2,2)$ supersymmetric gauge theories in two steps.

- Write the ground states as the Hilbert space formula, and (isomorphic-)map them to the magnon's configuration in the spin chain.
- Kinetic and dynamics of magnons map to the kinetic and dynamics of domain walls.

So the above similarities found by Nekrasov and Shatashvili between the integrable systems and the gauge theories are just a consequence of our framework.

Examples: Gr(k;N)

Now we focus on the simplest example: gauged linear sigma model for Gr(k;N). The vacuum equations are

$$(\sigma_a)^N = \Lambda^N.$$

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The Hamiltonian is simple:

$$H = \sum_i \mathcal{D}_i + \sum_i \bar{\mathcal{D}}_i.$$

The eigenstates are k-magnons

$$\mid k\rangle := A_k \sum_{1\le j_1<\cdots<j_k\le N} \prod_{a=1}^{k} \sum_{\mathcal{W}\in S_k} \left(\widehat{\sigma}_{\mathcal{W}(a)}\right)^{j_a} |j_1,\cdots,j_k\rangle_k.$$

The eigenvalue is

$$H\cdot \mid k\rangle := (e_1(\sigma) + e_1(\bar{\sigma})) \mid k\rangle$$

Quantum cohomology

One might have noticed that the holomorphic part of the eigenvalue $e_1(\sigma)$ is the cohomology generator of $H^{1,1}(Gr(k;N))$. One may expect that all of the generators can be constructed as the eigenvalues of the corresponding operators. It was first done by mathematicians (Postnikov '05, C. Korff and C. Stroppel '09) that one can construct a general elementary function:

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Open XX model

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All the discussions are similar to the closed one by only focusing on the (anti-)holomorphic part.

K-theoretic of XX model

One can lift gauged linear sigma models to the 3d CS-matter theories on space-time $\mathbb{R}^2 \times S^1$. Compared to the 2d theory, one has one more parameter to label the vacua and others: the so-called Chern-Simons levels. We only focus on the gauge Chern-Simons level $k_{U(k)}$. We split it into two factors: $k_{U(1)}$ and $k_{SU(k)}$.

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• If we choose (Kapustin-Willet '2013)

$$k_{U(1)} = k_{SU(k)} = -\frac{N}{2},$$

the vacuum equations are

$$(1 - x_a)^N = \widetilde{q}.$$

So the integrable system is almost identical to one for two-dimensional gauge theory with the only change being to replace σ by $1 - x$.

K-theoretic XX model

- If we choose (Jockers et.al, Ueda and Yoshida '2019, Gu et.al '20 and '22)

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the vacuum equations are

$$(1 - x_a)^N = \widetilde{q}\,\frac{(x_a)^k}{\prod_{b=1}^{k} x_b}.$$

There is "interaction factor" on the right hand side

$$S_{ab} = -\frac{x_a}{x_b}.$$

This factor means the magnons interacts each other. Since this factor of this case is not a pure phase, the spin chain of this case is always an open one regardless of what the value of $\widetilde{q}$ is.
This is because if we regard

$$1 - x_a := e^{ip_a}$$

The momentum p_a is not real.

State space

One can propose the interacting state space as follows:

$$A_k \sum_{1\leq j_1<\cdots<j_k\leq N} \sum_{\mathcal{W}\in S_k} S_{\mathcal{W}(a)} e^{\sum_{a=1}^{k} ip_{\mathcal{W}(a)} j_a} |j_1 < \cdots < j_k\rangle_k,$$

where A_k is an overall normalization factor, and $\sum_{\mathcal{W}\in S_k} S_{\mathcal{W}}$ is a multiplication of scattering factors S_{ab}. One can show that the proposed state space is consistent with the symmetries.

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where A_k is an overall normalization factor, and $\sum_{\mathcal{W}\in S_k} S_{\mathcal{W}}$ is a multiplication of scattering factors S_{ab}. One can show that the proposed state space is consistent with the symmetries. The fundamental Hamiltonian of this system is a complex one, which was first proposed in a slightly different context by Gorbounov and Korff '2014:

$$h=\sum_{i}^{N}\mathcal{D}_i-\sum_{|i_1-i_2| \mod N>1}^{N}\mathcal{D}_{i_1}\mathcal{D}_{i_2}+\sum_{|i_a-i_b| \mod N>1}^{N}\mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3}+\cdots.$$

The number of sites is N, so only finitely many terms act non-trivially, and the series therefore terminates.

Gr(2;N)

Let us check whether the ground state wave functions proposed in our paper for this case are indeed the eigenstates of the Hamiltonian. To shorten the notation, we denote the ground state as $|\,\omega\rangle_2 = \sum_{1\le j_1<j_2\le N} a(j_1,j_2)|j_1 < j_2\rangle_2$, where

$$a(j_1,j_2) = A\left(e^{i(p_1 j_1 + p_2 j_2)} + S_{21}e^{i(p_2 j_1 + p_1 j_2)}\right).$$

The boundary condition is $a(j_1, j_2 + N) = q \cdot a(j_2, j_1)$, which gives

$$e^{i(p_1 j_1 + p_2 j_2)}e^{ip_2 N} + S_{21}e^{ip_1 N}e^{i(p_2 j_1 + p_1 j_2)} = q\left(e^{i(p_1 j_2 + p_2 j_1)} + S_{21}e^{i(p_2 j_2 + p_1 j_1)}\right).$$

The vacuum equations can be read from the above as

$$e^{ip_1 N} = qS_{12}, \qquad e^{ip_2 N} = qS_{21}.$$

So it is consistent.

Gr(2;N)

To compute $H \mid \omega\rangle_2$, special care is needed when two overturned spins are sitting next to each other. We find

$$\begin{aligned} H \mid \omega\rangle_2 &= \sum_{1\leq j_1<j_2\leq N} \left(a(j_1+1,j_2)+a(j_1,j_2+1)-a(j_1+1,j_2+1)\right) \mid j_1<j_2\rangle_2 \\ &\quad - \sum_{1\leq j\leq N} \left(a(j+1,j+1)-a(j+1,j+2)\right) \mid j<j+1\rangle_2. \end{aligned}$$

In order to obey the eigenstate condition, the contact terms in the last line of the above equation should be vanishing:

$$a(j+1,j+1)-a(j+1,j+2)=0.$$

If we test the coefficient as $a(j_1,j_2)=Ce^{i(p_1j_1+p_2j_2)}+De^{i(p_1j_2+p_2j_1)}$, the vanishing contact terms all give

$$\frac{C}{D}=-\frac{x_1}{x_2}.$$

This is certainly consistent with our scattering factor S_{12} in the vacuum equations. The procedure applies to a general Grassmannian.

Geometric basis

The Hamiltonian actually has a geometrical meaning. See Buch and Mihalcea '08. The eigenvalue of this Hamiltonian is the first Schubert class of the $Gr(k; N)$:

$$H \mid \omega\rangle_k = \mathcal{O}_\square \mid \omega\rangle_k$$

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$$\mathcal{O}_{\square} := 1 - \prod_{a=1}^{k} x_a .$$

For example, if k=2 the factor in the above equation is

$$a(j_1+1, j_2) + a(j_1, j_2+1) - a(j_1+1, j_2+1) = (z_1 + z_2 - z_1 z_2) a(j_1, j_2)$$

where the coefficient $(z_1 + z_2 - z_1 z_2) = 1 - x_1 x_2$ (where $z = 1 - x$) is indeed the first Schubert class of $Gr(2,N)$. Thus, one may naturally expect that higher Schubert classes are the eigenvalues of the higher Hamiltonian as well. Quantum K-theory of $Gr(k;N)$ from the integrable model has been investigated by Gorbounov and Korff '2014, although their construction was based on a five-vertex model rather than a spin chain.

Seiberg(-like) Duality is P-symmetry

In gauge theories, there is a so-called Seiberg(-like) duality between two gauge theories:

$$Gr(k; N) \cong Gr(N-k; N).$$

The gauged linear sigma models for them are different: one is $U(k)$ gauge group and the other one is $U(N-k)$. However, the two gauge theories share the same vacuum structures and BPS spectrum. This relation is nontrivial in the sense that it is not a symmetry of a single gauge theory.

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But from the spin chain, it is a P-symmetry. To see this, we notice that the spin up/down maps to the spin down/up under the P symmetry.

$$P : | +\rangle \mapsto | -\rangle.$$

This says that the k-magnons map to the N-k magnons. This is exactly the Seiberg(-like) duality between two gauge theories.

A unification of gauge theories via spin chain

In the spin chain, we have spin operators: $S_{\pm}$. They have no field theoretic correspondence since they change the number of magnons which can not happen in a field theoretic process.

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In the spin chain, we have spin operators: $S_{\pm}$. They have no field theoretic correspondence since they change the number of magnons which can not happen in a field theoretic process. If we map the k-magnons of the spin chain to the U(k) gauge theory, then from the point of view of the spin chain. Different gauge theories can be connected via the spin operators. Maybe something deep behind this...

The Yang-Baxter equation

In the Yang-Baxter equation, the R-matrix is the fundamental variable used to describe the dynamics. It is defined by a map

$$R_{ab} : \mathcal{V}_a \otimes \mathcal{V}_b \mapsto \mathcal{V}_a \otimes \mathcal{V}_b.$$

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For the cases in this talk, the basis chosen for $\mathcal{V}$ is

$$|+,+\rangle, \quad |-,+\rangle, \quad |+,-\rangle, \quad |-,-\rangle.$$

Since the basis is transformed under the R-matrix, we expect that the off-diagonal entries of the R-matrix correspond to the (anti)-domain walls in our context.

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<u>Example:</u> For the Grassmannian, we have the R-matrix

$$\begin{pmatrix} a & & & \\ & b & c & \\ & \widetilde{c} & \widetilde{b} & \\ & & & \widetilde{a} \end{pmatrix}.$$

So two options for a generic $\widetilde{q}$: c or $\widetilde{c}$ is vanishing. They are the so-called five-vertex models.

R-matrix

For XXX or XXZ model, if the Hamiltonian is a hermitian operator, which says:

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For XYZ model, we also have domain wall configurations:

$$|-,-\rangle \mapsto |+,+\rangle, \quad \text{or} \quad |+,+\rangle \mapsto |-,-\rangle.$$

The R-matrix for this model should be enlarged as

$$\begin{pmatrix} a & & & d \\ & b & c & \\ & c & b & \\ d & & & a \end{pmatrix}.$$

One can derive the Yang-Baxter equation from the dynamic constraint of domain walls. See Gu '22.

A comment on the four-dimensional CS theory

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- if $C = \mathrm{Torus}$, it corresponds to XYZ model.

Recall that the spin operator, $S_z = \sum_i S_{z,i}$, is actually half of the axial-R symmetry operator

$$S_z = \frac{1}{2} F_A$$

If we regard the GLSM as an effective theory of 4d $\mathcal{N}{=}1$ gauge theory on $\Sigma \times C$, and the operator F_A depends on the sharp of C, which is exactly what the 4d-CS theory suggested. There should be more deep connections...

Bosonic chain

There are bosonic integrable systems.. For example, there is a q-deformed bosonic integrable system. The previous study for wave-function was using the algebraic Bethe ansatz. From quantum field theory, we can give a new anstatz and the q-deformed algebra will be emerged from this new ansatz. See Gu 2023.

Conclusion, chances and challenges

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- We showed how a spin chain emerges from the two-dimensional supersymmetric gauge theory. **Domain walls play a crucial role**.
- We discussed several examples.
- We talked about the Yang-Baxter equations with new insight, and commented on the four-dimensional CS theory.

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- We discussed several examples.
- We talked about the Yang-Baxter equations with new insight, and commented on the four-dimensional CS theory.

Chances and challenges

- Exact results from 2D supersymmetric gauge theory can help us understand quantum integrable systems, and vice versa.
- Many important ingredients of quantum integrable systems—such as the Yangian—have not yet been discussed about its correspondence in gauge theory.
- The relationships between different gauge theories that realize a given integrable system.
- Most importantly: how can we facilitate communication between different communities?

Thanks!