Polynomial algebras and solutions of a class of spin models

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Talk given at the International Workshop on "Challenges in Integrability", 9–21 Nov 2025, Wuhan

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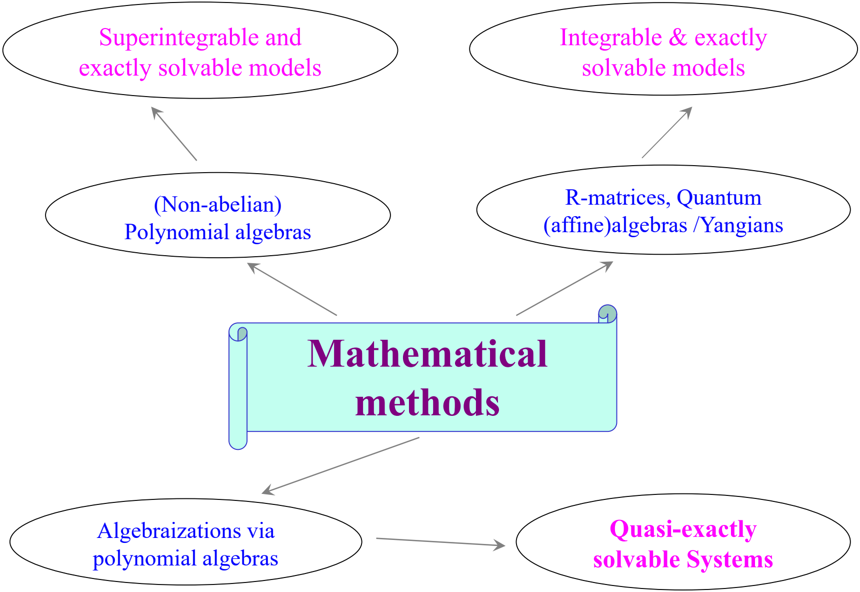
General overview



Quasi-exact solvability

In 1980-90's, systems occupying an intermediate place between exactly solvable and non-solvable ones, called quasi-exactly solvable systems (QES), were introduced (e.g. Turbiner - CMP 88, Gonzárez-López, Kamran & Olver - CMP 93, Ushveridze, QES models in quantum mechanics, IoP Publishing, 94, Sasaki &Takasaki - JPA 2001). These are non-trivial models and have found applications in a wide range of fields.

A system is called QES if only a finite number of eigenvalues and corresponding eigenfunctions can be obtained analytically and in a closed form while the remaining spectrum and wavefunctions of the system are not accessible by algebraic means.

Definition

(Turbiner, Phys. Report **642** (2016), 1-71) A linear operator H is called QES if it has a single finite-dimensional invariant subspace V_N with explicitly described basis $\{\xi_1, \xi_2, \ldots, \xi_{\dim V_N}\}$, that is

$$HV_N \subseteq V_N, \quad \dim V_N < \infty, \quad V_N = \mathrm{span}\left\{\xi_1, \xi_2, \ldots, \xi_{\dim V_N}\right\}.$$

An immediate consequence of this characterization of QES is that operator $\mathcal{H}$ can be diagonalized *algebraically* and closed-form expressions of spectra can be obtained in the (solvable) subspace $\mathcal{V}_{\mathcal{N}}$. But the remaining spectrum is not in general analytically accessible and can only be computed via approximation methods (though sometimes rather accurately).

Quasi-exact solvability vs. exact solvability

A quantum mechanical system is exactly solvable (ES) if all its eigenvalues and the corresponding eigenfunctions can be determined exactly and in a closed-form.

Consider an infinite set of finite-d linear spaces V_N with property

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_N \subset \cdots V,$$

where V is the completion space. Such an object is called flag V. If a linear operator H satisfies $H : V_N \to V_N$, $N = 0, 1, 2, \ldots$, then H is said to preserve the infinite flag V.

In terms of the characterizations of solvability of a linear operator (e.g. Hamiltonian of a quantum system) via the existence of invariant polynomial subspaces, have

Definition

A linear operator H is said to be exactly solvable (ES) if it preserves an infinite flag V of finite-d spaces V_N, $N = 0, 1, 2, \ldots$, whose bases admit explicit description/form, i.e. for $V_N = \mathrm{span}\left\{\xi_1, \xi_2, \ldots, \xi_{\dim V_N}\right\}$,

$$HV_N \subseteq V_N \subseteq V_{N+1}, \quad \dim V_N < \infty, \quad \forall\, N = 0, 1, 2, \ldots.$$

Algebraic approach

Algebraic structures underlying QES systems are Lie algebras (Turbiner - CMP 88) or non-linear deformations of Lie algebras (see next page).

Let $\mathcal{G}$ denote a Lie algebra or nonlinear deformation of a Lie algebra, with generators $\{J_\alpha,\ \alpha = 1, 2, \cdots, \mathrm{dim}\mathcal{G}\}$. Assume that $\mathcal{G}$ has a f.d. representation and $\mathcal{V}_\mathcal{N}$ is its representation space, i.e. $J_\alpha \mathcal{V}_\mathcal{N} \subset \mathcal{V}_\mathcal{N}$, $\alpha = 1, 2, \cdots, \mathrm{dim}\mathcal{G}$. Then obviously any operator $P(J_\alpha)$ made of J_α which is an element of the universal enveloping algebra $U(\mathcal{G})$ of $\mathcal{G}$ has $\mathcal{V}_\mathcal{N}$ as its invariant subspace $P(J_\alpha) : \mathcal{V}_\mathcal{N} \to \mathcal{V}_\mathcal{N}$.

Definition

If the invariant subspace of the QES operator coincides with a f.d. representation space of $\mathcal{G}$, then the QES operator is said to be $\mathcal{G}$-algebraic QES operator.

Definition

If space $\mathcal{V}_\mathcal{N}$ is irreducible f.d. representation space of $\mathcal{G}$, then the $\mathcal{G}$-algebraic QES operator can be expressed in terms of the generators of $\mathcal{G}$. In this case, one says that the QES operator admits an algebraization or has a hidden $\mathcal{G}$ algebra structure or symmetry.

Polynomial algebras

Polynomial algebras are (non-linear) polynomial deformations of Lie algebras. They are hidden algebraic symmetries of a range of QES systems in the sense that they provide algebraizaitions of the QES systems (see, e.g. Lee, Yang & Zhang - JPA 2010, Lee, Links & Zhang - Nonlinearity 2011, Li, Marquette & Zhang - arXiv:2502.13365).

Definition

A rank-1 polynomial algebra of degree r is a nonlinear (polynomial) deformation of $sl(2)$ or $su(1,1)$ whose generators $P_{\pm,0}$ obey the communication relations

$$[P_0, P_\pm] = \pm P_\pm, \qquad [P_+, P_-] = \Phi^{(r)}(P_0) = \sum_{i=0}^{r} c_i P_0^i, \qquad c_r \neq 0.$$

Coefficients c_i are constants or central elements of the algebra.

Casimir of the algebra

$$K = P_+P_- + \phi^{(r+1)}(P_0 - 1) = P_-P_+ + \phi^{(r+1)}(P_0),$$

where

$$\phi^{(r+1)}(P_0) - \phi^{(r+1)}(P_0 - 1) = \Phi^{(r)}(P_0).$$

$\phi^{(r+1)}(P_0)$ is a polynomial function in P_0 of degree $(r+1)$ and can be determined uniquely, up to an additive constant.

Spin models

We consider the following three spin models of physical interest:

Lipkin–Meshkov–Glick (LMG) model (Lipkin, Meshkov & Glick - NPB 1965):

$$\mathcal{H} = \Delta\, J_0 + g\,(J_+^2 + J_-^2).$$

Molecular asymmetric rigid rotor (King, Hainer & Cross - J. Chem. Phys. 1943):

$$\begin{aligned}\mathcal{H} &= aJ_x^2 + bJ_y^2 + cJ_z^2 \\ &= \frac{2c-a-b}{2}J_0^2 + \frac{a-b}{4}(J_+^2 + J_-^2) + \frac{a+b}{2}C, \qquad a \neq b,\end{aligned}$$

where C is the Casimir of $sl(2)$ which takes value $j(j+1)$ in spin-j representation.

Two-axis countertwisting squeezing model (Kitagawa & Ueda - PRL 1991):

$$H = \frac{\chi}{2i}(J_x J_y + J_y J_x) = \frac{\chi}{2i}(J_+^2 - J_-^2).$$

These models are special cases of a class of spin systems with Hamintonian

$$\mathcal{H} = c_+\, J_+^k + c_-\, J_-^k + \sum_{s=1}^{k} c_0\, J_0^s + c_*,$$

where c_+, c_-, c_s, c_* are model parameters so that $\mathcal{H}$ is hermitian.

Underlying polynomial algebra structure

The polynomial algebra defined by (Li, Marquette & Zhang - JSTAT 2025):

$$\begin{aligned} [P_0, P_\pm] &= \pm P_\pm, \\ [P_+, P_-] &= \phi^{(2k)}(P_0, C) - \phi^{(2k)}(P_0 - 1, C), \end{aligned}$$

where C is the Casimir operator of $sl(2)$ and $\phi^{(2k)}(P_0, C)$ is the polynomial of degree $2k$ in P_0 and C,

$$\phi^{(2k)}(P_0, C) = -\prod_{i=1}^{k}[C - (kP_0 + k - i + 1)(kP_0 + k - i)] + \prod_{i=1}^{k}[C - i(i-1)],$$

is a nonlinear deformation of $sl(2)$, to be denoted as $pl(sl(2))$. When $k = 1$ the above algebra reduces to $sl(2)$.

$pl(sl(2))$ has the $sl(2)$ **spin operator realization:**

$$P_\pm = J_\pm^k, \qquad P_0 = \frac{1}{k} J_0$$

and thus is the hidden algebraic structure underlying the spin models.

Representations of $pl(sl(2))$

$pl(sl(2))$ has $\min(k, 2j+1)$ lowest weight states, $|j,0;p\rangle \sim J_+^p|j,0\rangle$, labelled by quantum number $p = 0, 1, \cdots, \min(k-1, 2j)$. Here $|j,0\rangle$ is the lowest weight state of $sl(2)$. Then the irreducible $sl(2)$ module V_j is decomposed into the direct sum of irreducible $pl(sl(2))$ modules $V_{j,p}$, $V_j = \bigoplus_{p=0}^{\min(k-1,2j)} V_{j,p}$, where $V_{j,p}$ is spanned by $|j,m;p\rangle \sim J_+^{p+km}|j,0\rangle$. This gives the representations of $pl(sl(2))$,

$$P_+|j,m;p\rangle = \prod_{i=1}^{k} [(p+km+i)(2j-p+1-km-i)]^{\frac{1}{2}} |j,m+1;p\rangle,$$

$$P_-|j,m;p\rangle = \prod_{i=1}^{k} [(p+km-i+1)(2j-km-p+i)]^{\frac{1}{2}} |j,m-1;p\rangle,$$

$$P_0|j,m;p\rangle = \left(m + \frac{p-j}{k}\right) |j,m;p\rangle.$$

$P_+|j,\mathcal{N};p\rangle = 0$, $\quad \mathcal{N} \equiv \dfrac{2j-p-q}{k}$, where q is a non-negative integer taking specific values $q = 0, 1, \ldots, \min(k-1, 2j)$ according to j and p so that $\mathcal{N}$ is always a non-negative integer. Such an q always exists. Thus, $m = 0, 1, \ldots, \mathcal{N}$, and the above representation of $pl(sl(2))$ is $\mathcal{N}+1$ dimensional.

Differential operator realization of $pl(sl(2))$

Identifying $|j,0\rangle$ with z^0, we can map the state $|j,m;p\rangle$ to a monomial in variable z. Setting $x = z^k$ to simplify the monomial, the mapping takes the form

$$|j,m;p\rangle \quad \sim \quad x^{p/k} \times \frac{x^m}{\sqrt{(p+km)(2j-p-km)!}}.$$

The factor $x^{p/k}$ can be removed by a simple basis transformation (i.e. gauge transformation). **With respect to the transformed basis,** $\left\{ \frac{x^m}{\sqrt{(p+km)(2j-p-km)!}} \right\}$, the single variable differential operator realization of the $(\mathcal{N}+1)$-dimensional rep of $pl(sl(2))$ is given by

$$P_+ = x \prod_{i=1}^{k} \left(2j - p - i + 1 - kx\frac{d}{dx}\right),$$

$$P_- = x^{-1} \prod_{i=1}^{k} \left(p - i + 1 + kx\frac{d}{dx}\right),$$

$$P_0 = x\,\frac{d}{dx} + \frac{p-j}{k}.$$

Note that there are no x^{-1} terms in P_- since $\prod_{i=1}^{k}(p-i+1) = 0$ for all the allowed values of p and the differential operator realization is for the "gauged-transformed" generators $P_{\pm,0}$ with respect to the treansformed basis vectors.

Algebraization of the spin models

The spin model Hamiltonian admits a $pl(sl(2))$ algebraization

$$\mathcal{H} = c_+ P_+ + c_- P_- + \sum_{s=1}^{k} c_s \, (k P_0)^s + c_*.$$

That is the model possesses a hidden $pl(sl(2))$ polynomial algebra symmetry.

After a gauge transformation to the wavefunction $\psi(x)$ of the Hamilton equation $\mathcal{H}\psi(x) = E\psi(x)$,

$$\psi(x) = x^{p/k}\,\varphi(x), \quad p = 0, 1, \ldots, \min(k-1, 2j),$$

the Hamilton equation becomes $H\varphi(x) = E\varphi(x)$ with H being gauged-transformed k-th order differential operator (Li, Marquette & Zhang - JSTAT 25)

$$\begin{aligned} H &= x^{-p/k}\,\mathcal{H}\,x^{p/k} \\ &= c_+ x \prod_{i=1}^{k}\left(2j - p - i + 1 - kx\frac{d}{dx}\right) + c_- x^{-1} \prod_{i=1}^{k}\left(p - i + 1 + kx\frac{d}{dx}\right) \\ &\quad + \sum_{s=1}^{k} c_s \left(kx\frac{d}{dx} + p - j\right)^s + c_*, \qquad p = 0, 1, \ldots, \min(k-1, 2j). \end{aligned}$$

Quasi-exact solvability of H

For any $n \in \mathbb{Z}_+$,

$$Hx^n = x^{n+1} \prod_{i=1}^{k} (2j - p - i + 1 - kn) + \text{lower order terms}.$$

When $n = \mathcal{N}$, the first term on the right-hand side of the above equation disappears.

Thus H is QES as it has an invariant subspace $\mathcal{V} = \mathrm{span}(1, x, \ldots, x^{\mathcal{N}})$ and the Hamilton equation $H\varphi(x) = E\,\varphi(x)$ has polynomial solutions of the form

$$\varphi(x) = \prod_{i=1}^{\mathcal{N}} (x - x_i), \qquad \varphi(x) \equiv 1 \text{ for } \mathcal{N} = 0,$$

where $\{x_i | i = 1, 2, \ldots, \mathcal{N}\}$ are the roots of the polynomial to be determined.

Bethe ansatz equations

$$E = \frac{H\varphi}{\varphi} = \sum_{i=1}^{k} P_i(x)\, i! \sum_{\alpha_1<\alpha_2<\ldots\alpha_i}^{\mathcal{N}} \frac{i!}{(x - x_{\alpha_1})(x - x_{\alpha_2})...(x - x_{\alpha_i})} + P_0(x),$$

where $P_i(x)$ and $P_0(x)$ are polynomials in x obtained from the expansion of the products in the differential operator realization of H.

This leads to the algebraic equations for the roots $\{x_i\}$,

$$\sum_{i=2}^{k} \sum_{\alpha_1<\alpha_2<\ldots\alpha_{i-1}\neq\alpha}^{\mathcal{N}} \frac{P_i(x_\alpha)\, i!}{(x_\alpha - x_{\alpha_1})(x_\alpha - x_{\alpha_2})...(x_\alpha - x_{\alpha_{i-1}})} + P_1(x_\alpha) = 0,$$

where $\alpha = 1, 2, \ldots, \mathcal{N}$. These equations are called Bethe ansatz equations.

The wavefunction $\varphi(x)$ becomes the eigenfunction of H in the space $\mathcal{V}_{\mathcal{N}}$ provided that the roots $\{\alpha_i\}$ of the polynomial $\varphi(x)$ are the solutions of the Bethe ansatz equations.

The quasi-exact energy eigenvalues are given by

$$E = c_* + \sum_{s=1}^{k} c_s \, (k\mathcal{N} + p - j)^s - c_+ \left[\prod_{i=1}^{k} (2j - p - i + 1 - k(\mathcal{N} - 1)) \right] \sum_{i=1}^{\mathcal{N}} x_i,$$

where x_i satisfy the Bethe ansatz equations.

The corresponding wave function of the system is

$$\psi(x) = x^{p/k} \, \varphi(x) = x^{p/k} \prod_{i=1}^{\mathcal{N}} (x - x_i), \qquad p = 0, 1, \ldots, \min(k-1, 2j).$$

In terms of the variable z (recalling $x = z^k$), the wave function takes the form

$$\psi(z) = z^p \, \varphi(z) = z^p \prod_{i=1}^{\mathcal{N}} (z^k - x_i), \qquad p = 0, 1, \ldots, \min(k-1, 2j).$$

Here $\{x_i\}$ are determined from the Bethe ansatz equations.

Values of p and q are constrained such that $\mathcal{N} = \dfrac{2j - p - q}{k}$ is a non-negative integer.

Example: LMG model

The Hamiltonian of the LMG model admits an algebraization in terms of $pl(sl(2))$ of degree 3 corresponding to the $k = 2$ case of the general results. That is, the LMG model Hamiltonian has a cubic algebra as its hidden symmetry. We have $p = 0, 1$ and

$$\mathcal{N} = j - \frac{p+q}{2}$$

with $q = 0, 1$ so that $\mathcal{N}$ is a non-negative integer. In terms of differential realization of the cubic algebra, the gauge-transformed version H takes the form of a 2nd-order differential operator, and the corresponding Schrödinger equation is

$$\begin{aligned} 4g(x^3+x)\frac{d^2\varphi}{dx^2} + [2g(1+2p) + 2\Delta x + 2g(3-4j+2p)x^2]\frac{d\varphi}{dx} \\ + \{[2gj(2j-1) + gp(1-4j+p)]x - [E + (j-p)\Delta]\}\,\varphi = 0. \end{aligned}$$

Solutions of this equation is given by

$$\varphi(x) = \prod_{i=1}^{\mathcal{N}}(x - x_i), \qquad \varphi(x) \equiv 1 \text{ for } \mathcal{N} = 0,$$

where $\{x_i\}$ are the roots of Bethe ansatz equations

$$\sum_{\ell \neq i}^{\mathcal{N}} \frac{2}{x_\ell - x_i} + \frac{g(3-4j+2p)x_i^2 + \Delta x_i + g(1+2p)}{2g(x_i^3 + x_i)} = 0, \quad i = 1, 2, \cdots, \mathcal{N}.$$

The corresponding energy eigenvalue E is

$$E = (2\mathcal{N} + p - j)\,\Delta - g(2j - p + 1 - 2\mathcal{N})(2j - p + 2 - 2\mathcal{N}) \sum_{i=1}^{\mathcal{N}} x_i .$$

From the constraint for $\mathcal{N}$,

$$j = \mathcal{N} + \frac{p+q}{2}, \quad \mathcal{N} \in \mathbb{Z}_+; \quad p = 0, 1; \quad q = 0, 1.$$

This gives rise to the following four cases:

$$\begin{aligned} &j \text{ even}: \quad p = q = 0 \quad \text{or} \quad p = 1 = q, \\ &j \text{ odd}: \quad p = 0,\ q = 1 \quad \text{or} \quad p = 1,\ q = 0, \end{aligned}$$

.

We now include the gauge factor and apply $x = z^2$ to obtain the wavefunction in terms of the variable z for the LMG model,

$$\psi(z) = z^p \prod_{i=1}^{\mathcal{N}} (z^2 - x_i), \qquad p = 0, 1,$$

where $p = 0$ and $p = 1$ correspond to even and odd sectors, respectively.

As examples, we consider $\mathcal{N} = 0$ and $\mathcal{N} = 1$ cases and present the corresponding energies and wavefunctions in the even and odd sectors of the model.

$\mathcal{N} = 0$ **Case**:

The energy and wavefunction are given by $E = (p - j)\,\Delta$ and $\psi(z) = z^p$, respectively. Here p (which takes the value 0 or 1) and the spin j are constrained so that $\mathcal{N} = j - (p + q)/2$, where $q = 0, 1$, is equal to 0. In this case, the energies and wavefunctions in the even sectors ($p = 0$) are

$$E_{even} = 0, \qquad \psi_{even}(z) = 1,$$

$$E_{even} = -\frac{\Delta}{2}, \qquad \psi_{even}(z) = 1,$$

which correspond to $j = 0$ (integer) and $j = 1/2$ (half-integer), respectively. The energies and wavefunctions in the odd sector ($p = 1$) are given by

$$E_{odd} = 0, \qquad \psi_{odd}(z) = z,$$

$$E_{odd} = \frac{\Delta}{2}, \qquad \psi_{odd}(z) = z,$$

associated with $j = 1$ (integer) and $j = 1/2$ (half-integer), respectively.

$\mathcal{N} = 1$ **case**:

There are two sets of energies and wavefunctions in the the even sector ($p = 0$),

$$E_{even} = \mp\sqrt{\Delta^2 + 4g^2}, \qquad \psi_{even}(z) = z^2 - \frac{\Delta \pm \sqrt{\Delta^2 + 4g^2}}{2g},$$

$$E_{even} = -\frac{\Delta}{2} \mp \sqrt{\Delta^2 + 12g^2}, \qquad \psi_{even}(z) = z^2 - \frac{\Delta \pm \sqrt{\Delta^2 + 12g^2}}{6g}$$

which correspond to $j = 1$ (integer) and $j = 3/2$ (half-integer), respectively. Similarly, the energies and wavefunctions in the odd sector ($p = 1$) are given by

$$E_{odd} = \mp\sqrt{\Delta^2 + 36g^2}, \qquad \psi_{odd}(z) = z^3 - \frac{\Delta \pm \sqrt{\Delta^2 + 36g^2}}{6g} z,$$

$$E_{odd} = \frac{\Delta}{2} \mp \sqrt{\Delta^2 + 12g^2}, \qquad \psi_{odd}(z) = z^3 - \frac{\Delta \pm \sqrt{\Delta^2 + 12g^2}}{2g} z,$$

associated with $j = 2$ (integer) and $j = 3/2$ (half-integer), respectively. (Note that other energies and wavefunctions for $j = 2$ are obtained from $\mathcal{N} = 2$ solutions.)

Summary and outlook

- The sipn models are different from usual Yang-Baxter integrable quantum spin chains. For instance, the underlying algebraic structures for the formers are polynomial algebras, while for the latters they are quantum (affine) algebras or Yangians.
- Algebraizations of spin models in terms of their hidden polynomial algebraic structures is the key for their exact solutions.
- Derivation of closed-form solutions is based on differential realizations of hidden symmetry algebras and and the Analytic/Functional Bethe Ansatz.
- QISM or Algebraic Bethe Ansatz method for such spin systems do not seem possible due to the lack of knowledge of R-matrix etc, in constrast to Yang-Baxter integrable spin chain models.
- Interesting open problems: Are polynomial algebras Hopf algebra and quasi-triangular? Affinizations of polynomial algebras? .....