Exact solutions of two solvable models: from two-body mass-imbalance system to Bose-Hubbard model with unidirectional hopping

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Nov. 20 2025

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Y Liu, F Qi, Y Zhang, S Chen, IScience. 22, 181 (2019).
M Zheng, Y Qiao, Y Wang, J Cao, S Chen, Phys. Rev. Lett. 132, 086502 (2024).

Outline

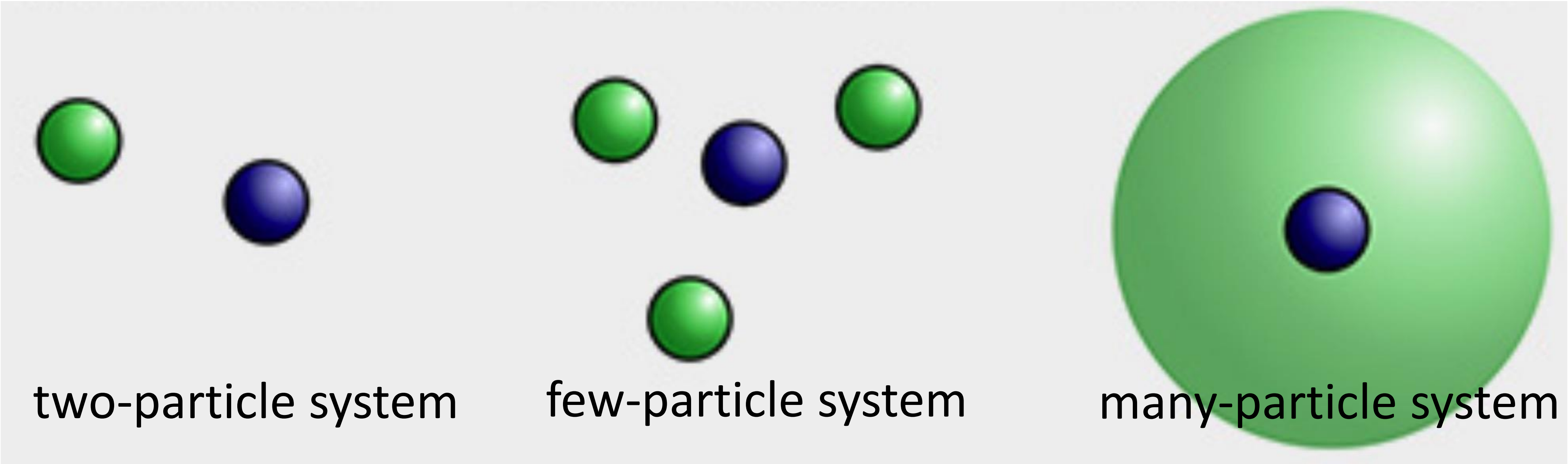
- Exact solution for mass-imbalance two-body system
- Bose-Hubbard model with unidirectional hopping

Background

More (three) is different！

Few-body problem is non-trivial.

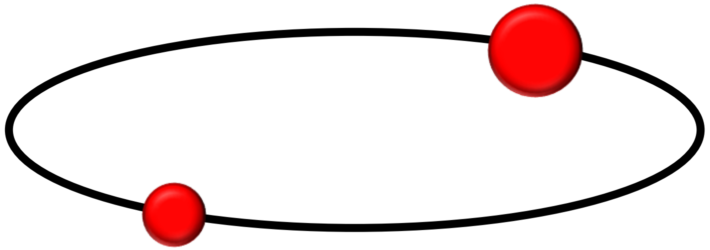
Studying the few-body systems allows us to better understand the many-body systems, e.g. Tan's relations



Exact solution to two-body system

The Hamiltonian is

$$H = -\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + g\delta(x_1 - x_2)$$



The periodic **boundary condition**

$$\Psi(x_1 + L, x_2) = \Psi(x_1, x_2 + L) = \Psi(x_1, x_2)$$

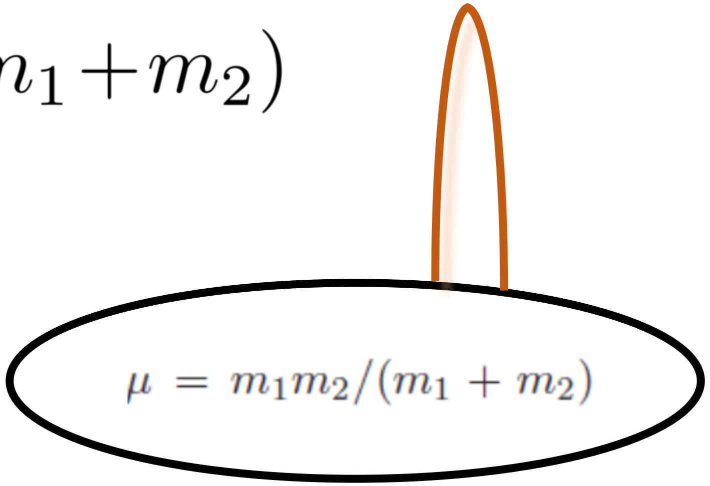
$$M = m_1 + m_2$$

Introducing relative and center-of-mass coordinates

$$x = x_1 - x_2 \qquad X = (m_1 x_1 + m_2 x_2)/(m_1 + m_2)$$

The Hamiltonian can be rewritten as

$$\hat{H} = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial X^2} \boxed{-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} + g\delta(x)}$$



The wavefunction can take the form of $\Psi(X,x) = e^{iKX}\varphi(x)$

with
$$\begin{aligned}\varphi(x) &= (A_+e^{ikx} + B_+e^{-ikx})\theta(x) \\ &+(A_-e^{ikx} + B_-e^{-ikx})\theta(-x)\end{aligned}$$

The spectra $\boxed{E = \frac{\hbar^2}{2M}K^2 + \frac{\hbar^2}{2\mu}k^2}$

$$K = k_1 + k_2$$
$$k = (m_2k_1 - m_1k_2)/(m_1+m_2)$$

Wavefunction in terms of x1 and x2 takes the form of

$$\boxed{\begin{aligned}\Psi(x_1,x_2) = &(A_+e^{ik_1x_1+ik_2x_2} + B_+e^{ik_1'x_1+ik_2'x_2})\theta(x_1-x_2) \\ &+ (A_-e^{ik_1x_1+ik_2x_2} + B_-e^{ik_1'x_1+ik_2'x_2})\theta(x_2-x_1)\end{aligned}}$$

where $k_1 = Km_1/(m_1+m_2)+k$, $k_2 = Km_2/(m_1+m_2)-k$, $k_1' = Km_1/(m_1+m_2)-k$, and $k_2' = Km_2/(m_1+m_2)+k$.

The periodic boundary condition gives

$$A_+ e^{ik_1 L} = A_-, \tag{9}$$

$$B_+ e^{ik_1' L} = B_-, \tag{10}$$

$$A_- e^{ik_2 L} = A_+, \tag{11}$$

$$B_- e^{ik_2' L} = B_+. \tag{12}$$

Mutiplying Eq.9 and 11 or Eq.10 and 12 gives rise to $e^{iKL} = 1$

So we get

$$\boxed{KL = 2\pi n}$$

n takes integer numbers

The continuous condition of wavefunction requires

$$A_{+} + B_{+} = A_{-} + B_{-}. \tag{6}$$

The connection condition at the x=0 gives rise to

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial}{\partial x}\varphi(x)_{x=\epsilon} - \frac{\partial}{\partial x}\varphi(x)_{x=-\epsilon}\right) + g\varphi(0) = 0,$$

$$\Downarrow$$

$$\boxed{ik\left(A_{+} - B_{+} - A_{-} + B_{-}\right) = \frac{\mu}{\hbar^2} g\left(A_{+} + B_{+} + A_{-} + B_{-}\right).} \tag{7}$$

We have six paramters to be determined. Now we have six equations already,
so we can determine all the parameters from these equations.

Combining Eqs 9, 10 and 6, we can represent B+, A-, B- in terms of A+. Substituting them into Eq.7, we can derive

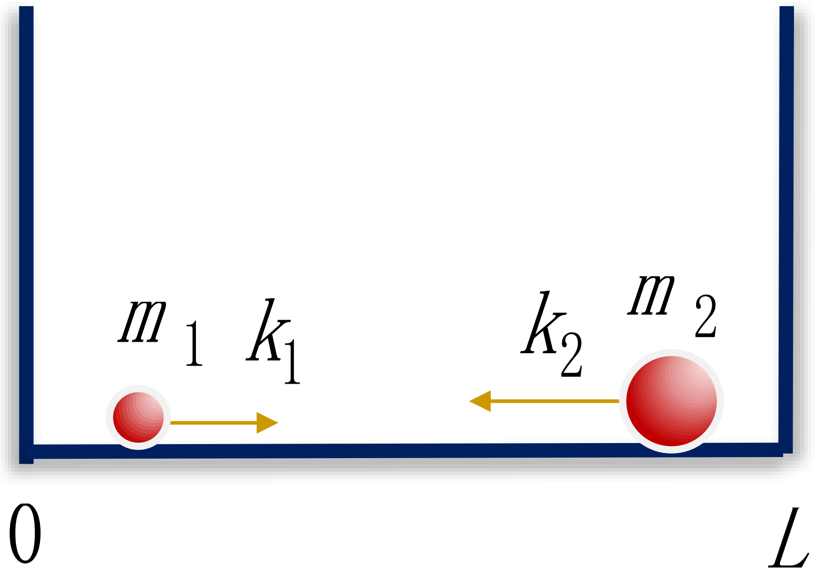
$$\boxed{k\frac{\hbar^2}{\mu g} = \frac{\sin(kL)}{\cos\left(K\frac{m_1 L}{m_1+m_2}\right) - \cos(kL)}}$$

When the mass ratio equals to 1, we can recover the Bethe-ansatz equations of two partice Lieb-Liniger model.

Xing Chen, Liming Guan, Shu Chen, Eur. Phys. J. D 64, 459 (2011)

We have solved the problem under **PBC**. But the story for the solution under **open boundary condition** is more complex.

The model with two mass-imbalanced interacting particles under OBC



Is there exact solution for the system with any mass ratio?

Nonergodicity condition (classical collision)

There are two types of collision processes, namely, the scattering between particles and the reflection process when the particle hits the wall.

The momentums of two particles before and after the collision

$$\mathbf{k} = (k_1, k_2)^T \qquad \mathbf{k}' = (k_1', k_2')^T$$

Total momentum and total energy of the particles are conserved

$$k_1 + k_2 = k_1' + k_2'$$

$$\frac{k_1^2}{2m_1} + \frac{k_2^2}{2m_2} = \frac{{k_1'}^2}{2m_1} + \frac{{k_2'}^2}{2m_2}$$

$$\frac{k_1^2 - k_1'^2}{m_1} = \frac{k_2'^2 - k_2^2}{m_2}$$

$$\frac{k_1 + k_1'}{k_2 + k_2'} = \eta$$

$$\eta = m_1/m_2$$

Nonergodicity condition (classical collision)

$$\mathbf{k}' = (k_1', k_2')^T \qquad \mathbf{k} = (k_1, k_2)^T$$

$$\mathbf{k}' = s\mathbf{k} \qquad \boxed{s(\eta) = \begin{pmatrix} \frac{\eta-1}{\eta+1} & \frac{2\eta}{\eta+1} \\ \frac{2}{\eta+1} & \frac{1-\eta}{\eta+1} \end{pmatrix}} \qquad \eta = m_1/m_2$$

The transformation matrix fulfills following relations:

$$\boxed{s(\eta)^2 = 1} \qquad \boxed{s(1/\eta) = \sigma_x s(\eta) \sigma_x}$$

Reflection $\mathbf{k}' = \pm\sigma_z \mathbf{k}$

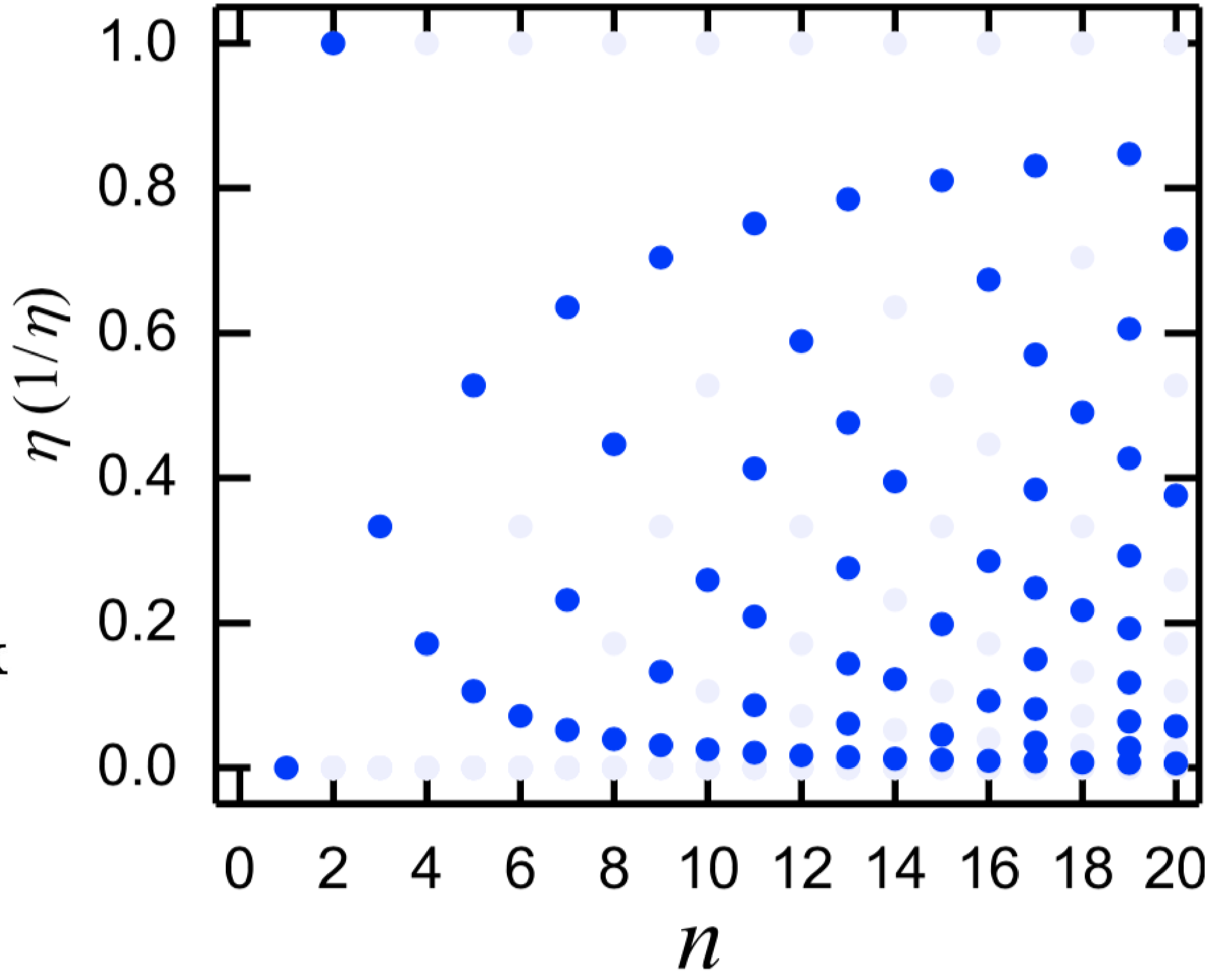
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The scattering and reflection always occur alternately

$$\mathbf{k}' = s(\eta)(-\sigma_z)s(\eta)\sigma_z s(\eta)\sigma_z \mathbf{k}$$

$$(s(\eta)\sigma_z)^n \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \pm \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

$$(\sigma_z s(\eta))^n \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \pm \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

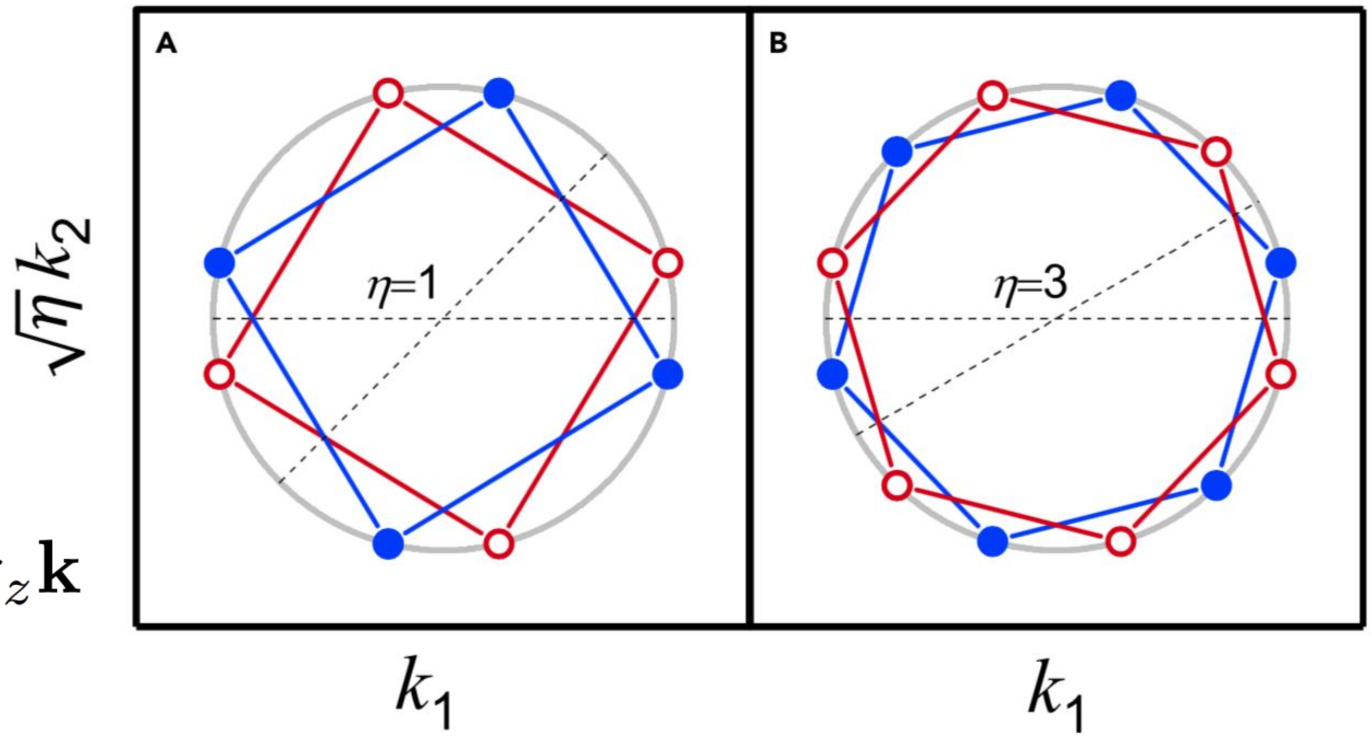


The condition

$$\boxed{\eta = \tan^2 l\pi/2n} \qquad 1 \leq l \leq n$$

In the case of equal mass

$$\eta = \tan^2 \pi/4 = 1$$



The full momentum set

$$\mathbf{k}, r\mathbf{k}, r^2\mathbf{k}, r^3\mathbf{k}, \sigma_z\mathbf{k}, r\sigma_z\mathbf{k}, r^2\sigma_z\mathbf{k}, r^3\sigma_z\mathbf{k}$$

with $r = s(1)\,\sigma_z$

The first nontrivial case $\eta = \tan^2 \pi/3 = 3$

The full momentum set

$$R = UrU^{-1} = \begin{pmatrix} \cos\frac{m\pi}{n} & -\sin\frac{m\pi}{n} \\ \sin\frac{m\pi}{n} & \cos\frac{m\pi}{n} \end{pmatrix}$$

$$\mathbf{k}, r\mathbf{k}, r^2\mathbf{k}, \cdots, r^5\mathbf{k}, \sigma_z\mathbf{k}, r\sigma_z\mathbf{k}, r^2\sigma_z\mathbf{k}, \cdots, r^5\sigma_z\mathbf{k}$$

with $r = s(3)\,\sigma_z$

Momentum determined by D_{2n} group

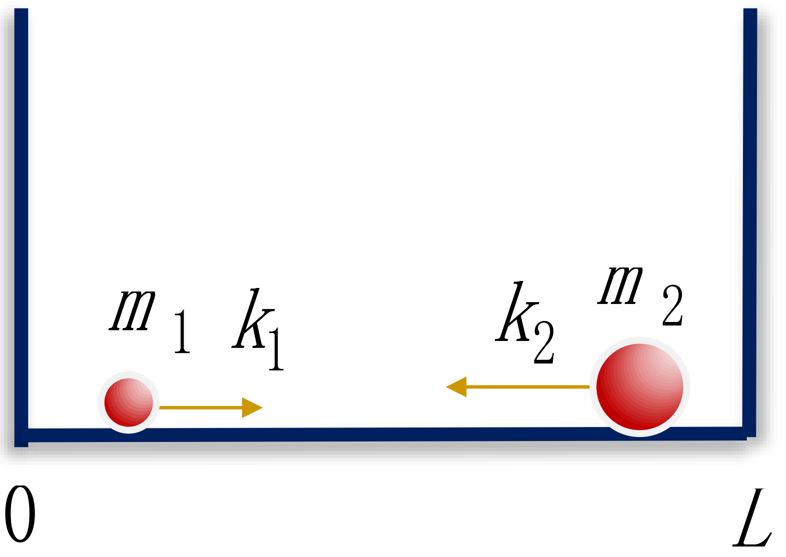
The relationship between mass ratio η and the dihedral group.

$\boxed{\eta = \tan^2 l\pi/2n}$

η	n	l	the dihedral group
$0, +\infty$	1	1	D_2
1	2	1	D_4
$1/3$	3	1	D_6
3	3	2	D_6
$3-2\sqrt{2}$	4	1	D_8
$3+2\sqrt{2}$	4	3	D_8
$1-2/\sqrt{5}$	5	1	D_{10}
$5-2\sqrt{5}$	5	2	D_{10}
$1+2/\sqrt{5}$	5	3	D_{10}
$5+2\sqrt{5}$	5	4	D_{10}
$7-4\sqrt{3}$	6	1	D_{12}
$7+4\sqrt{3}$	6	5	D_{12}
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The Hamiltonian can be written as

$$H = -\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + g\delta(x_1 - x_2)$$



The open boundary condition

$$\Psi(x_i = \pm L/2) = 0$$

The wave function is taken as the Bethe Ansatz type

$$\Psi(x_1, x_2) = \theta(x_2 < x_1)\sum_j A_{j+} e^{i(d_j \mathbf{k})^T \cdot \mathbf{x}}$$

$$+ \theta(x_1 < x_2)\sum_j A_{j-} e^{i(d_j \mathbf{k})^T \cdot \mathbf{x}},$$

$$D_{2n} = \{R, \sigma_z | R^{2n} = \sigma_z^2 = 1, \sigma_z R \sigma_z^{-1} = R^{-1}\}$$

$$d_j \in D_6$$

Y X Liu, F Qi, Y B Zhang, S Chen, **iScience** 22, 181-194(2019)

Boundary equations

The reflection matrix for particle 1 and 2

$$R_j\left(1,\pm\right)=\frac{A_{j\pm}}{A_{\underline{j}\pm}}=-\exp\left[\mp iL\sum_{l=1,2}d_j^{1l}k_l\right]\qquad R_j\left(2,\pm\right)=\frac{A_{j\pm}}{A_{\bar{j}\pm}}=-\exp\left[\pm iL\sum_{l=1,2}d_j^{2l}k_l\right]$$

The coefficients	Corresponding quasimomentums	The relation
$A_{j\pm}$	$(k_1^{\prime},k_2^{\prime}) = (d_j\mathbf{k})^T$	
$A_{\underline{j}\pm}$	$(-k_1^{\prime},k_2^{\prime}) = \left(d_{\underline{j}}\mathbf{k}\right)^T$	$d_{\underline{j}} = -\sigma_z d_j$
$A_{\bar{j}\pm}$	$(k_1^{\prime},-k_2^{\prime}) = \left(d_{\bar{j}}\mathbf{k}\right)^T$	$d_{\bar{j}} = \sigma_z d_j$

The scattering between two particles

$$\left[\frac{\partial\Psi}{\partial x}|_{x=0_+} - \frac{\partial\Psi}{\partial x}|_{x=0_-}\right] - \frac{2\mu}{\hbar^2} g\Psi|_{x=0} = 0 \Rightarrow$$

$$\begin{aligned} & i\left(\frac{d_j^{11}k_1 + d_j^{12}k_2}{1+\eta} - \frac{d_j^{21}k_1 + d_j^{22}k_2}{1+1/\eta}\right) \\ & \times \left(A_{j+} - A_{k+} - A_{j-} + A_{k-}\right) \\ = \quad & \frac{2\mu}{\hbar^2} g\left(A_{j-} + A_{k-}\right), \end{aligned}$$

The coefficients	Corresponding quasimomentums	The relation
$A_{j\pm}$	$(d_j\mathbf{k})^T$	$d_j = s d_k$
$A_{k\pm}$	$(d_k\mathbf{k})^T$	

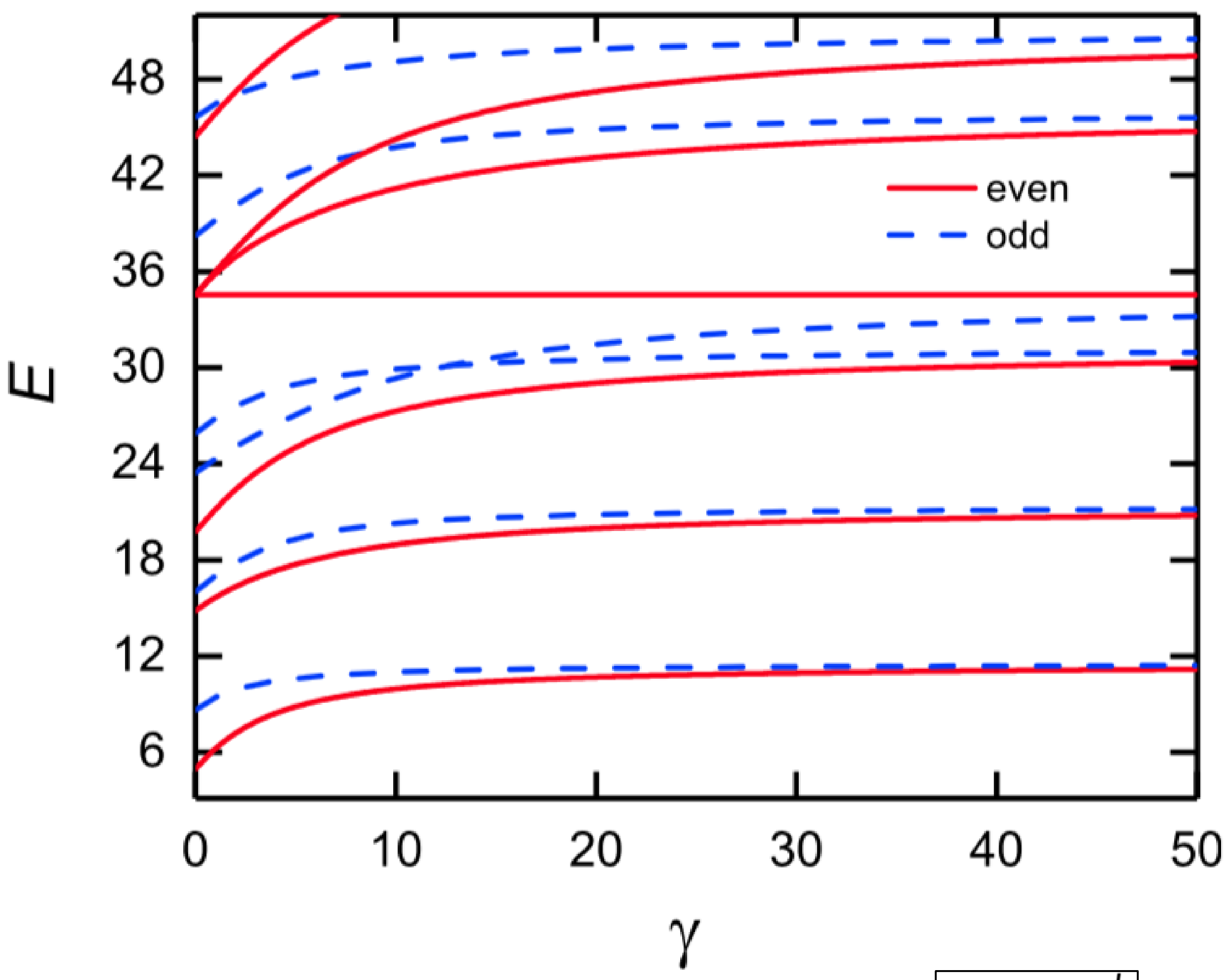
Bethe-type-ansatz equations for the system with $\eta = \tan^2 \pi/3 = 3$

$$\begin{cases} k_1 + 3k_2 - \dfrac{2\mu}{\hbar^2} g\left(\cot\dfrac{(k_1+k_2)L}{2} + \cot k_2 L\right) = 0, \\ k_1 - 3k_2 - \dfrac{2\mu}{\hbar^2} g\left(\cot\dfrac{(k_1-k_2)L}{2} - \cot k_2 L\right) = 0, \end{cases}$$

$$\begin{cases} k_1 + 3k_2 + \dfrac{2\mu}{\hbar^2} g\left(\tan\dfrac{(k_1+k_2)L}{2} + \tan k_2 L\right) = 0, \\ k_1 - 3k_2 + \dfrac{2\mu}{\hbar^2} g\left(\tan\dfrac{(k_1-k_2)L}{2} - \tan k_2 L\right) = 0. \end{cases}$$

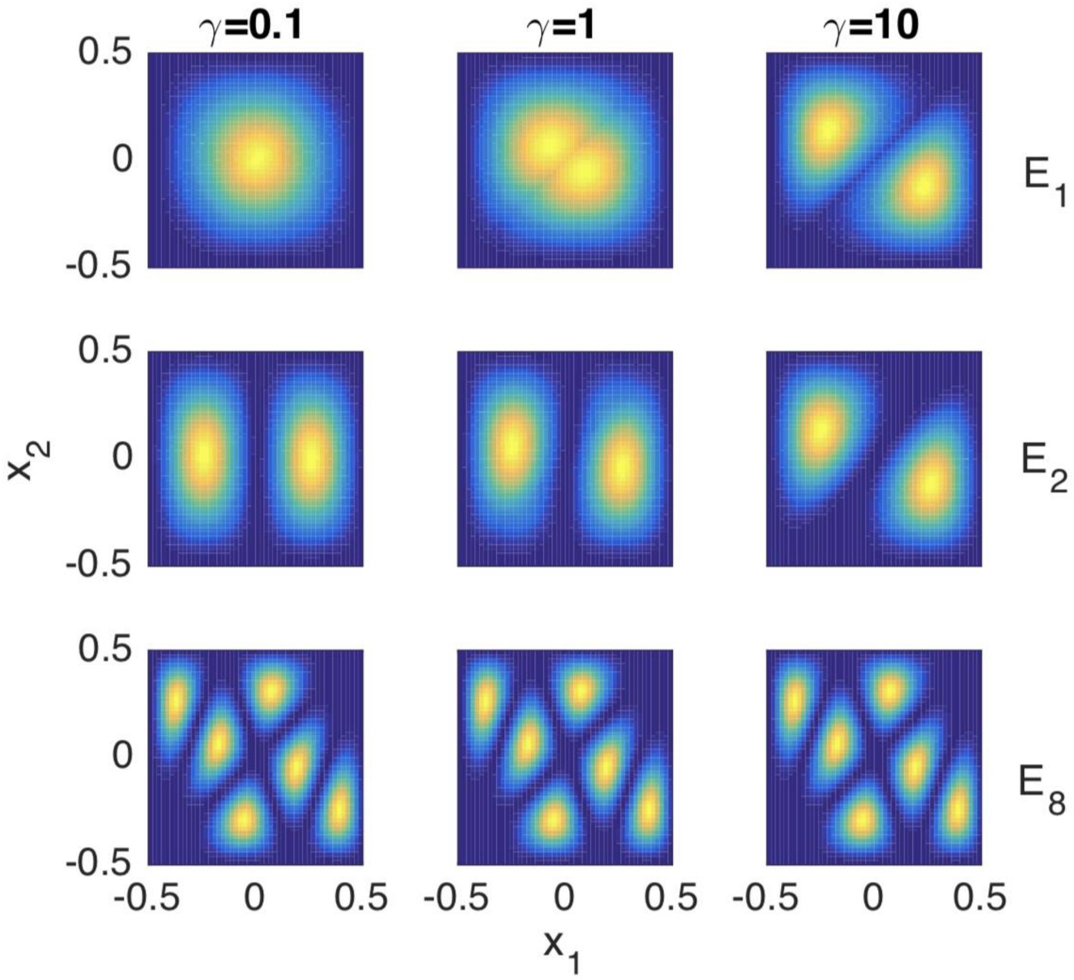
$$\mu = m_1/(\eta + 1)$$

The spectra



$$\gamma = \frac{\mu g L}{\hbar^2}$$

The wavefunctions



$$\Psi(x_1, x_2) = \pm\Psi(-x_1, -x_2)$$

The hard-core limit: $g = \infty$

The wave function satisfies the boundary condition

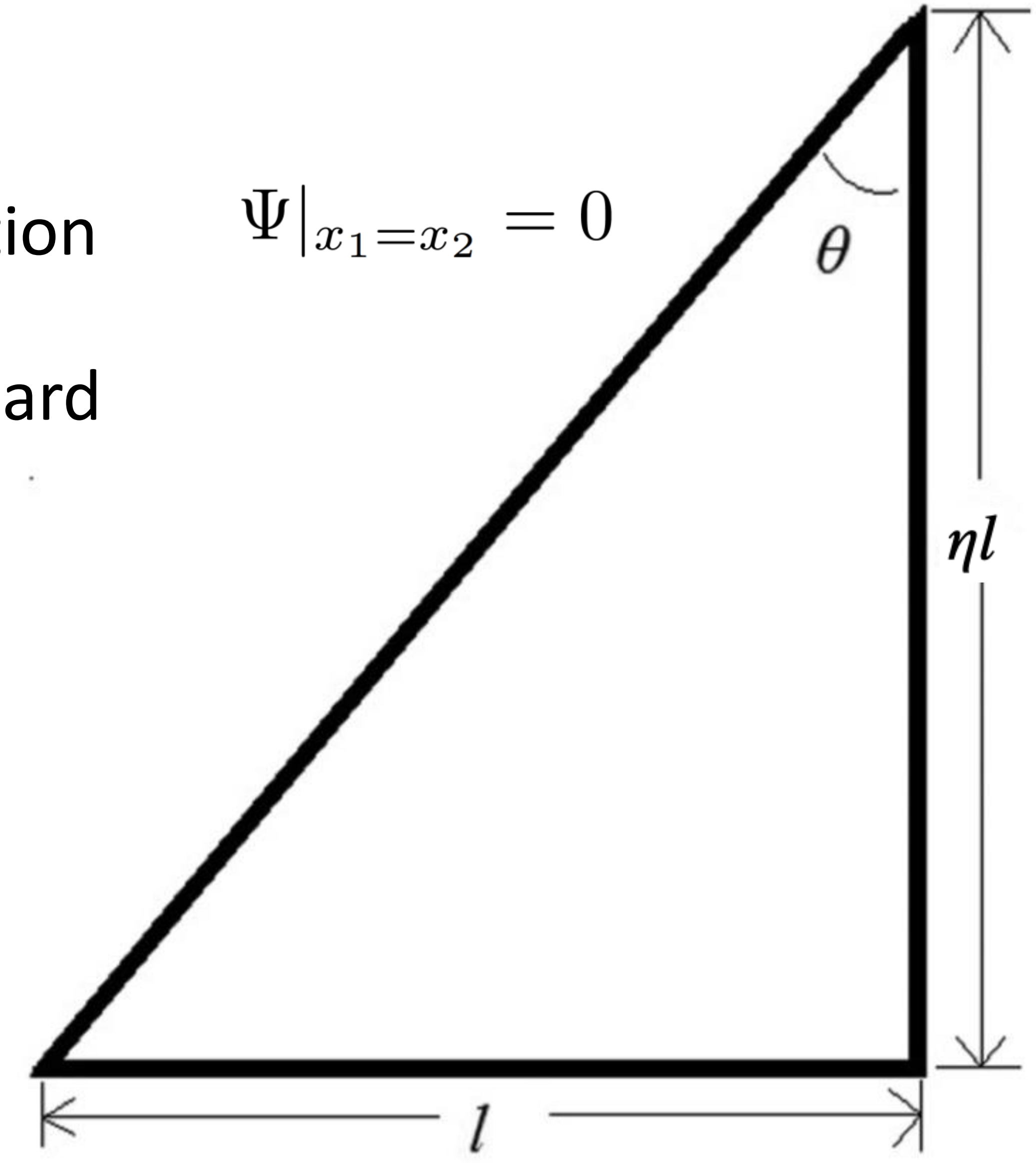
Mapping to one particle in a right triangular billiard

The system is integrable only when

$\eta = \tan^2 \pi/4 = 1$ or $\eta = \tan^2 \pi/3 = 3$

P. Richens and M. Berry, Physica D: Nonlin. Phenom. 2, 495(1981)

Biao Wu et al. use the exact diagonalization method
for arbitrary η



D. Zhang, H.T. Quan, B. Wu. PRE 94, 022150 (2016) arXiv: 2311.15155, SciPost Phy Core, 7,062 (2024)

SciPost Physics Core Submission

Asymmetric Bethe Ansatz

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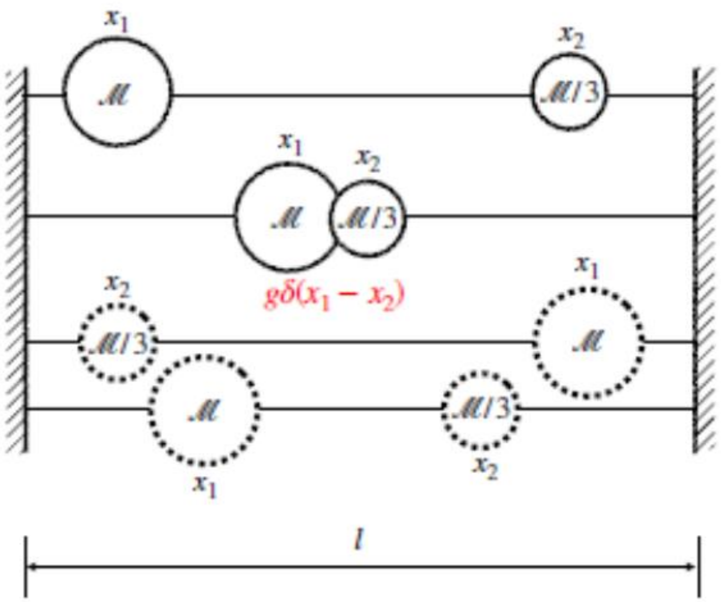
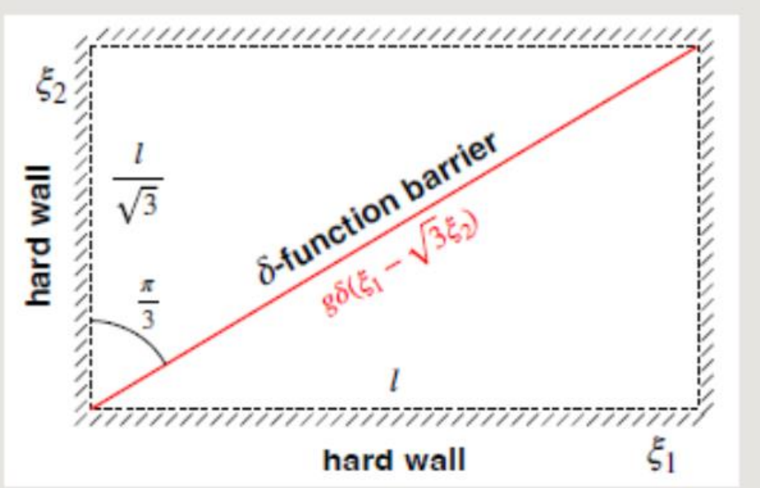
The recently proposed exact quantum solution for two δ-function-interacting particles with a mass-ratio 3 : 1 in a hard-wall box [Y. Liu, F. Qi, Y. Zhang and S. Chen, iScience 22, 181 (2019)] violates the conventional necessary condition for a Bethe Ansatz integrability, the condition being that the system must be reducible to a superposition of semi-transparent mirrors that is invariant under all the reflections it generates. In this article, we found a way to relax this condition: some of the semi-transparent mirrors of a known self-invariant mirror superposition can be replaced by the perfectly reflecting ones, thus breaking the self-invariance. The proposed name for the method is *Asymmetric Bethe Ansatz* (Asymmetric BA). As a worked example, we study in detail the bound states of the nominally non-integrable system comprised of a bosonic dimer in a δ-well. Finally, we show that the exact solution of the Liu-Qi-Zhang-Chen problem is a particular instance of the the Asymmetric BA.

Inspiration: Unexplained integrability the Liu-Qi-Zhang-Chen system





SciPost Physics Core Submission

Kaleidoscope Yang-Baxter Equations in the Study of Gaudin's Kaleidoscope models

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Abstract

Recently, M. Oshanii and other researchers have proposed the method of Asymmetric Bethe Ansatz, which serves as a theoretical tool to extend the solvable range of the Bethe Ansatz by "breaking" part of the mirror symmetry (through the introduction of a completely reflecting boundary). Within this framework, the integrability conditions originally introduced by Gaudin have been generalized. In this work, we undertake a detailed investigation of the relationship between D_{2N} symmetry and integrability based on Gaudin's generalized kaleidoscope model. We demonstrate that the mathematical essence of integrability in this class of models is characterized by a newly proposed Kaleidoscope Yang-Baxter equation. Furthermore, we point out that whether a model can be solved by the coordinate Bethe Ansatz is determined not only by the consistency relations satisfied by the scattering matrices, but also by the boundary conditions of the model and the symmetry subspace in which the solution is sought. Through numerical studies based on the finite element method, we confirm that Bethe Ansatz integrability emerges in specific symmetry sectors. In addition, we explore novel quantum algebraic identities arising from the Kaleidoscope Yang-Baxter equation. Our work thus constitutes an important refinement of the Asymmetric Bethe Ansatz approach.

Contents

Yicong Yu, Xiwen Guan, et.al Under preparation

Bose-Hubbard model

Exact solution of 1D **Hubbard model** provides benchmarks for understanding many many-body physical phenomena.

$$H = -t \sum_{\langle i,j \rangle}^{N} b_i^{\dagger} b_j + \frac{U}{2} \sum_{i}^{N} n_i \left(n_i - 1\right) - \mu \sum_{i}^{N} n_i .$$

The one-dimensional **Bose Hubbard model** is **not integrable**.

Choy and Haldane, **Phys. Lett. A**, 90, 83 (1982)

Constructing new integrable model with simple form and clear physical meanings is a fascinating mission!

Spin XXZ model (Bethe),
Lieb-Liniger model, Yang-Gaudin model, Fermi Hubbard model (Lieb-Wu)
Kondo model

Bose-Hubbard model with unidirectional hopping solution

T**he Bose Hubbard model with unidirectional hopping** (UHBH) is described by

$$H = -t[\sum_{j=1}^{N-1} b_j^\dagger b_{j+1} + \epsilon b_N^\dagger b_1] + \frac{U}{2}\sum_{j=1}^{N} n_j(n_j - 1)$$

The model is non-Hermitian as it only includes hopping terms along one direction.

In the framework in the quantum inverse scattering method, we can construct the model and prove its integrability.

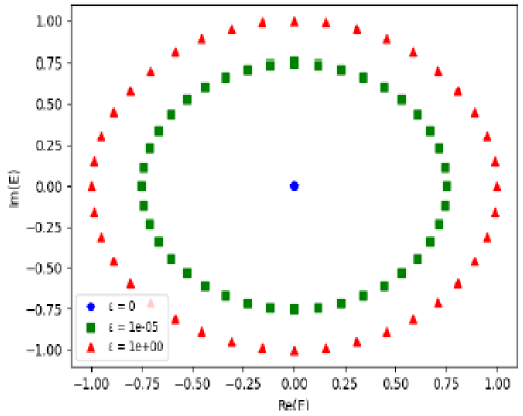
M Zheng, Y Qiao, Y Wang, J Cao, **S. Chen**, Phys, Rev. Lett. 132, 086502 (2024)

Unidirectional hopping model

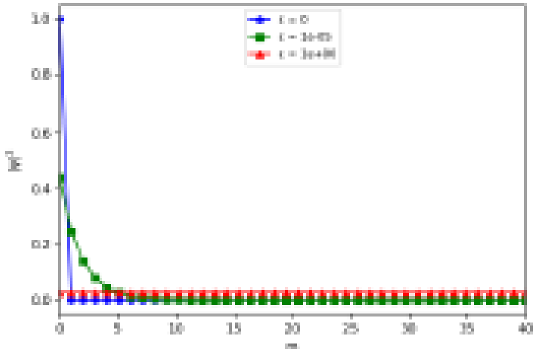
When U=0, the model is a special case of Hanato-Nelson model with unidirectional hopping

$$\boxed{E^N - \epsilon = 0}$$

$$E/t = \epsilon^{\frac{1}{N}} e^{i\theta}, (\theta = \frac{2m\pi}{N}, m = 1, 2...N)$$



$$\boxed{|\psi\rangle = c_0[1, \epsilon^{\frac{1}{N}} e^{i\theta}, \epsilon^{\frac{2}{N}} e^{i2\theta} \ldots \epsilon^{\frac{N-1}{N}} e^{i(N-1)\theta}]^T}$$



The wavefuntion shows the **skin effect** as it always accumulate on one of the boundary

Integrability and Yang-Baxter equation

The integrability of our model can be guaranteed by the **Yang-Baxter equation**.

$$R_{0,\bar{0}}(u-v)L_{0,j}(u)L_{\bar{0},j}(v) = L_{\bar{0},j}(v)L_{0,j}(u)R_{0,\bar{0}}(u-v)$$

$$R_{0,\bar{0}}(u) = \begin{pmatrix} u-1 & 0 & 0 & 0 \\ 0 & u & -1 & 0 \\ 0 & -1 & u & 0 \\ 0 & 0 & 0 & u-1 \end{pmatrix} \qquad L_{0,j}(u) = \begin{pmatrix} u-n_j & gb_j \\ gb_j^\dagger & -g^2 \end{pmatrix}$$

Our Hamiltonian can be derived from the transfer matrix

$$T_0(u) = L_{0,N}(u)L_{0,N-1}(u)\cdots L_{0,1}(u)K_0. \quad K_0 = diag(1,\epsilon).$$

where u is the spectral parameter and tr_0 means the trace in the two-dimensional auxiliary space. the subscripts $\{1,\dots,N\}$ denote the physical spaces and the Lax operator $L_{0,j}$ and K_0 can be expressed in the auxiliary space as

Integrability and Yang-Baxter equation

It can be proven that T satisfies the Yang-Baxter equation

$$R_{0,\bar{0}}(u-v)T_0(u)T_{\bar{0}}(v) = T_{\bar{0}}(v)T_0(u)R_{0,\bar{0}}(u-v).$$

$$\boxed{\begin{aligned} t(u) &= tr_0[L_{0,N}(u)L_{0,N-1}(u)\cdots L_{0,1}(u)K_0] \\ &= u^N + C_1 u^{N-1} + C_2 u^{N-2} + \cdots . \end{aligned}}$$

$$C_1 = -\sum_{j=1}^{N} n_j,$$

$$C_2 = \frac{1}{2}\sum_{i\neq j}^{N} n_i n_j + g^2 \left(\sum_{j=1}^{N-1} b_j^\dagger b_{j+1} + \epsilon b_N^\dagger b_1 \right)$$

From Yang-Baxter equation, we get $[t(u), t(v)] = 0$.

The expansion coefficients Cn are commutative and can be used to construct the conserved quantities

Model and its exact solution

The UHBH Hamiltonian can be construct as

$$H = \frac{t}{2g^2}\left(C_1^2 + C_1\right) - \frac{t}{g^2} C_2 \qquad [H, C_n] = 0.$$

Since an infinite number of conserved quantities can be obtained, H of UHBH is integrable.

$$H = -t\Big[\sum_{j=1}^{N-1} b_j^\dagger b_{j+1} + \epsilon b_N^\dagger b_1\Big] + \frac{U}{2}\sum_{j=1}^{N} n_j(n_j - 1), \qquad U = t/g^2.$$

Diagonalization of H by using Algebraic Bethe ansatz method
(Quantum inverse scattering method)

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

Diagonalization of t matrix

$$t(u) = A(u) + D(u)$$

From the RTT relation, we get

$$\begin{aligned}
&[A(u), A(v)] = [B(u), B(v)] = [C(u), C(v)] = [D(u), D(v)] = 0, \\
&A(u)C(v) = \frac{u-v-1}{u-v} C(v)A(u) + \frac{1}{u-v} C(u)A(v), \\
&D(u)C(v) = \frac{u-v+1}{u-v} C(v)D(u) - \frac{1}{u-v} C(u)D(v).
\end{aligned}$$

Reference state is chosen as product of vacuum states on each site

$$|0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_N .$$

$$\begin{aligned}
A(u)\,|0\rangle &= u^N\,|0\rangle\,, \\
D(u)\,|0\rangle &= \epsilon(-\frac{t}{U})^N\,|0\rangle\,, \\
B(u)\,|0\rangle &= 0.
\end{aligned}$$

Diagonalization of t matrix

The Bethe state is constructed by

$$|\mu_1, ..., \mu_M\rangle = \prod_{j=1}^{M} C(\mu_j)\,|0\rangle$$

By using the commuting relations between C and A (D), we get

$$\begin{aligned}
A(u)C(\mu_1)...C(\mu_M) &= \prod_{j=1}^{M} \frac{u-\mu_j-1}{u-\mu_j} C(\mu_1)...C(\mu_M)A(u) \\
&+ \sum_{j=1}^{M} \frac{1}{u-\mu_j} \prod_{l\neq j}^{M} \frac{\mu_j-\mu_l-1}{\mu_j-\mu_l} C(\mu_1)...C(\mu_{j-1})C(u)C(\mu_{j+1})...C(\mu_M)A(\mu_j), \\
D(u)C(\mu_1)...C(\mu_M) &= \prod_{j=1}^{M} \frac{u-\mu_j+1}{u-\mu_j} C(\mu_1)...C(\mu_M)D(u) \\
&- \sum_{j=1}^{M} \frac{1}{u-\mu_j} \prod_{l\neq j}^{M} \frac{\mu_j-\mu_l+1}{\mu_j-\mu_l} C(\mu_1)...C(\mu_{j-1})C(u)C(\mu_{j+1})...C(\mu_M)D(\mu_j).
\end{aligned}$$

Diagonalization of t matrix

Then it follows

$$t(u)\,|\mu_1,...,\mu_M\rangle = (A\,(u)+D\,(u))\prod_{j=1}^{M}C(\mu_j)\,|0\rangle$$

$$=\Lambda(u)\prod_{j=1}^{M}C(\mu_j)\,|0\rangle+\sum_{j=1}^{M}\Lambda_j(u)C(\mu_1)...C(\mu_{j-1})C(u)C(\mu_{j+1})...C(\mu_M)\,|0\rangle\,,$$

where

$$\Lambda(u)=u^N\prod_{j=1}^{M}\frac{u-\mu_j-1}{u-\mu_j}+\epsilon(-\frac{t}{U})^N\prod_{j=1}^{M}\frac{u-\mu_j+1}{u-\mu_j},$$

$$\Lambda_j(u)=\frac{1}{u-\mu_j}\left(\mu_j^N\prod_{l\neq j}^{M}\frac{\mu_j-\mu_l-1}{\mu_j-\mu_l}-\epsilon(-\frac{t}{U})^N\prod_{l\neq j}^{M}\frac{\mu_j-\mu_l+1}{\mu_j-\mu_l}\right).$$

Diagonalization of t matrix

Let the unwanted term vanish, we get

$$\Lambda_j(u) = 0. \longrightarrow \boxed{\frac{U^N}{(-t)^N}\mu_j^N = \epsilon\prod_{l\neq j}^{M}\frac{\mu_j-\mu_l+1}{\mu_j-\mu_l-1}, \quad j=1,\cdots,M.}$$

$\Lambda(u)$ is the eigenvalue of t matrix

Expanding $\Lambda(u)$ and comparing with the polynomial of $t(u)$, we can get

$$C_1\left|\mu_1,...,\mu_M\right\rangle = -M\left|\mu_1,...,\mu_M\right\rangle,$$
$$C_2\left|\mu_1,...,\mu_M\right\rangle = \left(-\sum_{j=1}^{M}\mu_j + \frac{M(M-1)}{2}\right)\left|\mu_1,...,\mu_M\right\rangle.$$

Model and its exact solution

Using the algebraic Bethe ansatz method, we get the eigenvalues of UHBH model is

$$E = \sum_{j=1}^{M} \beta_j$$

$$\beta_j = U\mu_j$$

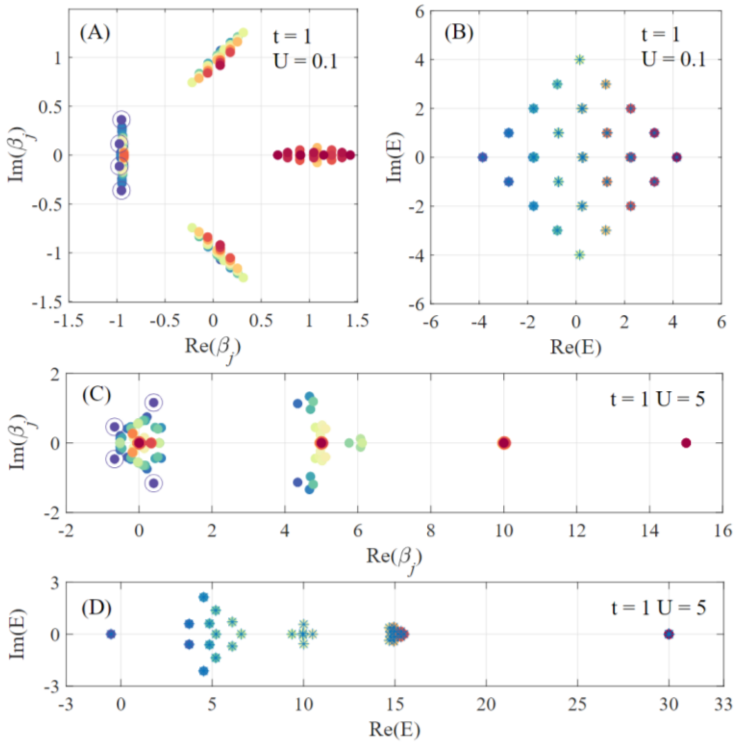
where the M Bethe roots $\{\beta_j\}$ should satisfy the **Bethe ansatz equations** (BAEs)

$$\boxed{\left(\frac{\beta_j}{-t}\right)^N = \epsilon \prod_{l\neq j}^{M} \frac{\beta_j - \beta_l + U}{\beta_j - \beta_l - U}, \quad j = 1, \cdots, M}$$

Taking the logarithm of the above Eq., we get

$$N \ln\left(\frac{\beta_j}{-t\,\sqrt[N]{\epsilon}}\right) = \sum_{l\neq j}^{M} \Theta\left(\frac{U}{\beta_j - \beta_l}\right) + 2i\pi I_j, \qquad \Theta(z) \equiv 2\,\mathrm{arctanh}(z)$$

Solutions of BAEs



When U is small,

$$\beta_j \approx t e^{\frac{i\pi(N+2l)}{N}}, l = 1, \ldots, N, j = 1, \ldots, M$$

When U is large,

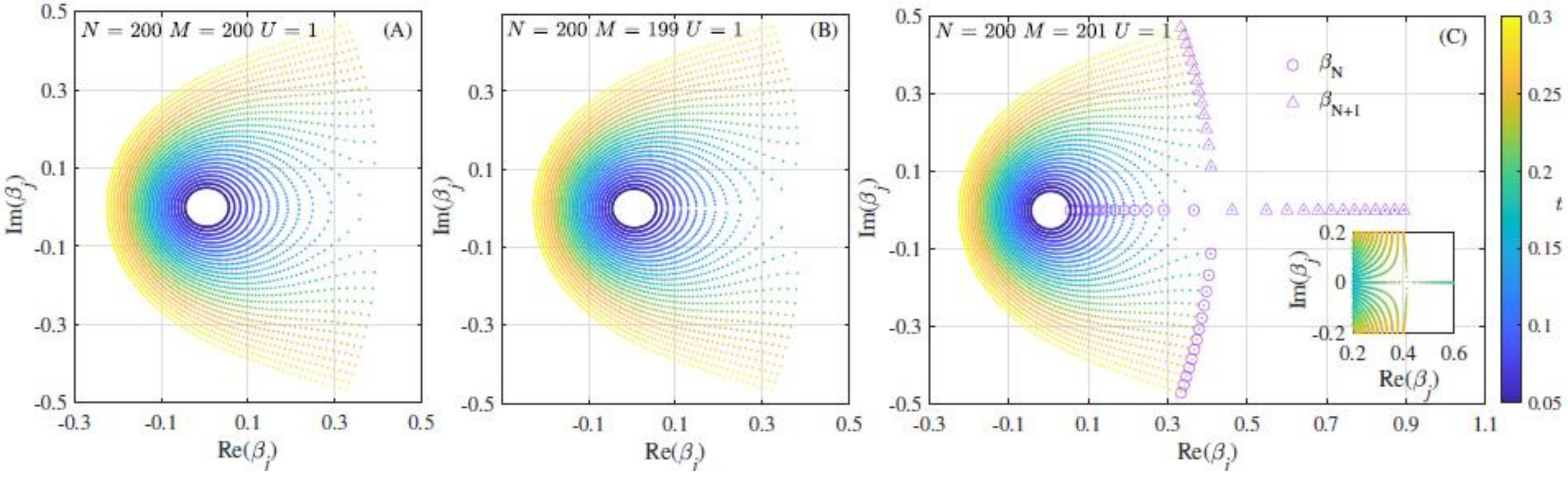
$$|\beta_j / U| \approx k, k = 1, \ldots, M-1$$

The ground state is defined by the minimum of real parts of the eigenvalues.

The solution of BAEs with $N = 4$ and $M = 4$

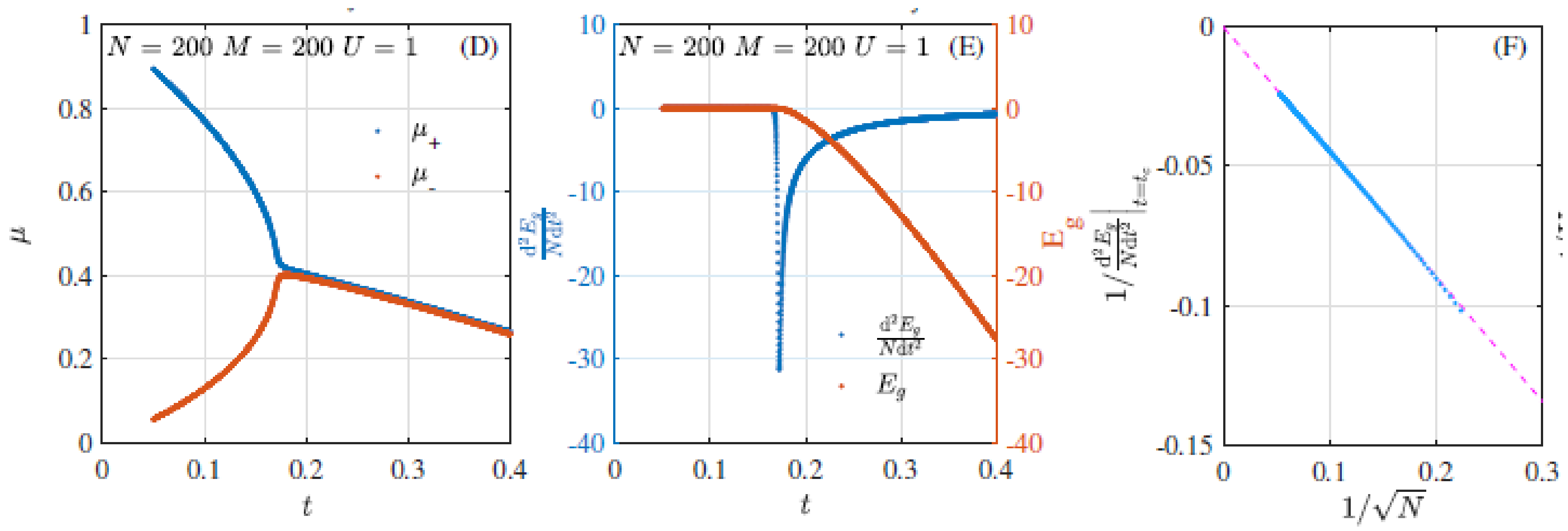
The superfluid-Mott transition

$$\rho = \frac{M}{N} = 1$$



The ground state Bethe roots patterns of BAEs

The superfluid-Mott transition



$$\mu_+ = E_N(\rho N + 1) - E_N(\rho N)$$

$$\mu_- = E_N(\rho N) - E_N(\rho N - 1)$$

In the thermodynamic limit

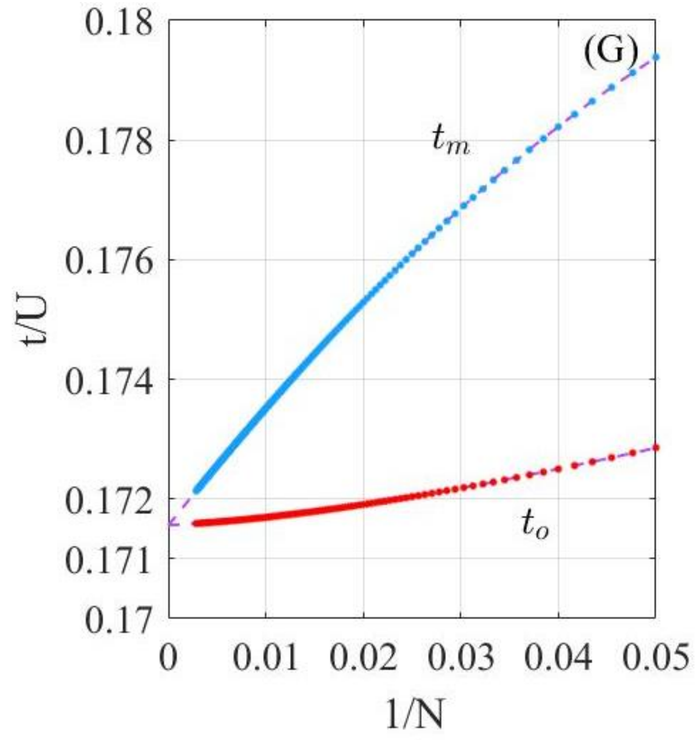
$$\mu_+ = \mathrm{Re}(\beta_{N+1}), \quad \mu_- = \mathrm{Re}(\beta_N).$$

When $t < t_o$, $\Delta\mu = \mathrm{Re}(\beta_{N+1}) - \mathrm{Re}(\beta_N) > 0.$

When $t > t_o$, $\Delta\mu = \mathrm{Re}(\beta_{N+1}) - \mathrm{Re}(\beta_N) = 0.$

t_o is the position that gap opens

The superfluid-Mott transition

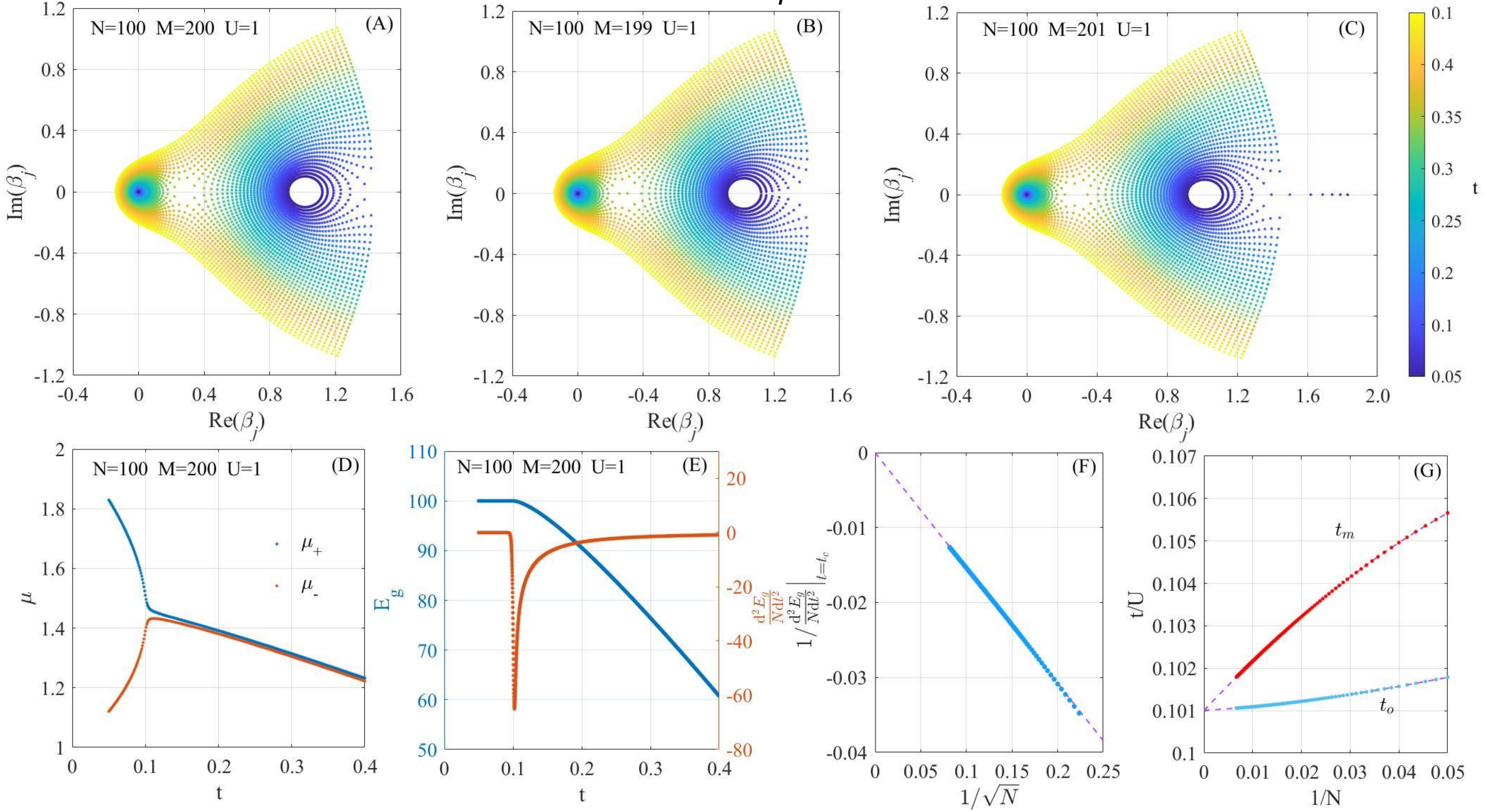


From a scaling, in the thermodynamic limit, the value of t_o and t_m is the same ($|t_o - t_m| < 10^{-4}$).

The value of the superfluid-Mott transition critical point is $t_c = 0.17155 \pm 10^{-4}$.

The superfluid-Mott transition

The result of $\rho = 2$



Non-Hermitian skin effect vs interaction

Using the coordinate Bethe ansatz method, the eigenstates of the system can be written out explicitly

$$|\Psi\rangle_m = \sum_{x_1,\cdots,x_M=1}^{N} \psi(x_1,\cdots,x_M) b^\dagger_{x_1}\cdots b^\dagger_{x_M}|0\rangle$$

where

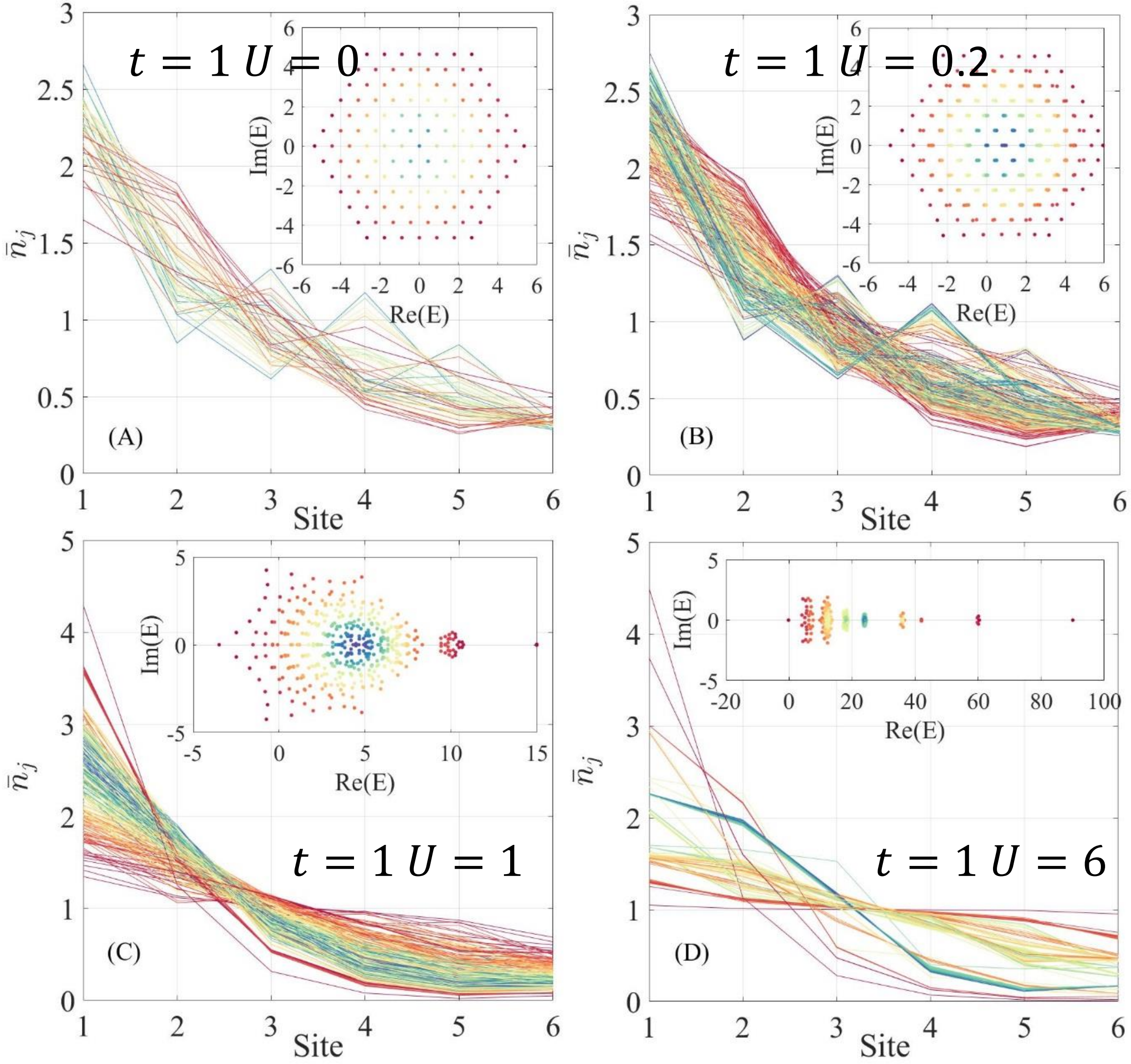
$$\psi = \sum_{p,q} A_p(q) \prod_{j=1}^{M} \left(\frac{\beta_{p_j}}{-t}\right)^{x_{q_j}} \theta(x_{q_1} \leq \cdots \leq x_{q_M})$$

The amplitude A_p satisfies

$$\frac{A_{p_1\cdots p_{j+1}p_j\cdots p_M}}{A_{p_1\cdots p_j p_{j+1}\cdots p_M}} = \frac{\beta_{p_{j+1}} - \beta_{p_j} + U}{\beta_{p_{j+1}} - \beta_{p_j} - U}$$

Each set of Bethe roots $\{\beta_j\}$ of BAEs completely determine one eigenstate.

Fate of non-Hermitian skin effect



Multiplying the M BAEs

$$\left(\frac{\beta_j}{-t}\right)^N = \epsilon \prod_{l\neq j}^{M} \frac{\beta_j - \beta_l + U}{\beta_j - \beta_l - U}, \quad j = 1, \cdots, M$$

we have

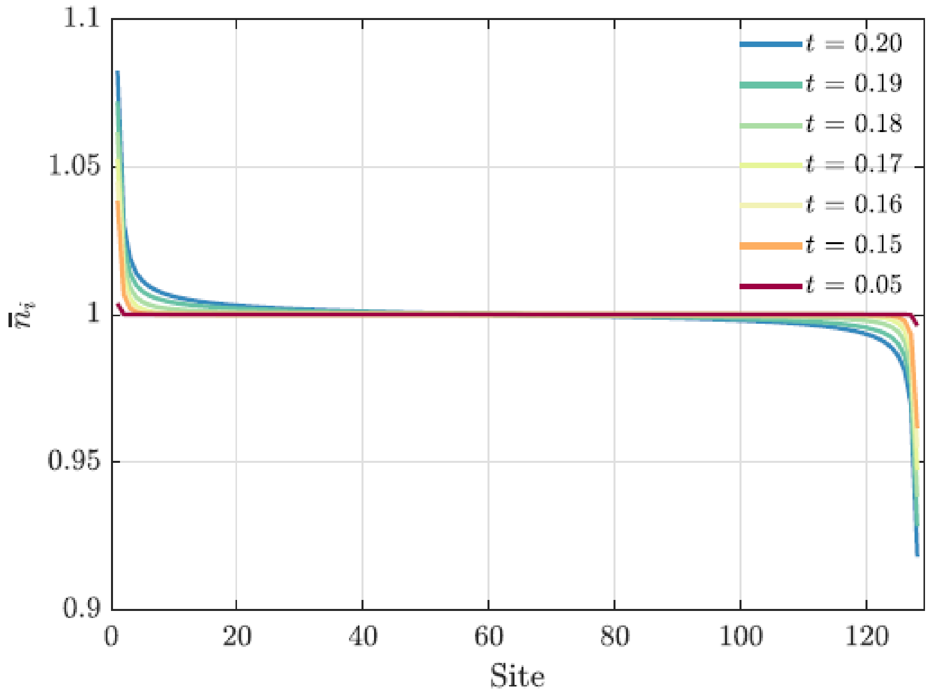
$$\prod_{j=1}^{M} \left|\frac{\beta_j}{t}\right| = \epsilon^{M/N}$$

which gives that ($x_j < N$)

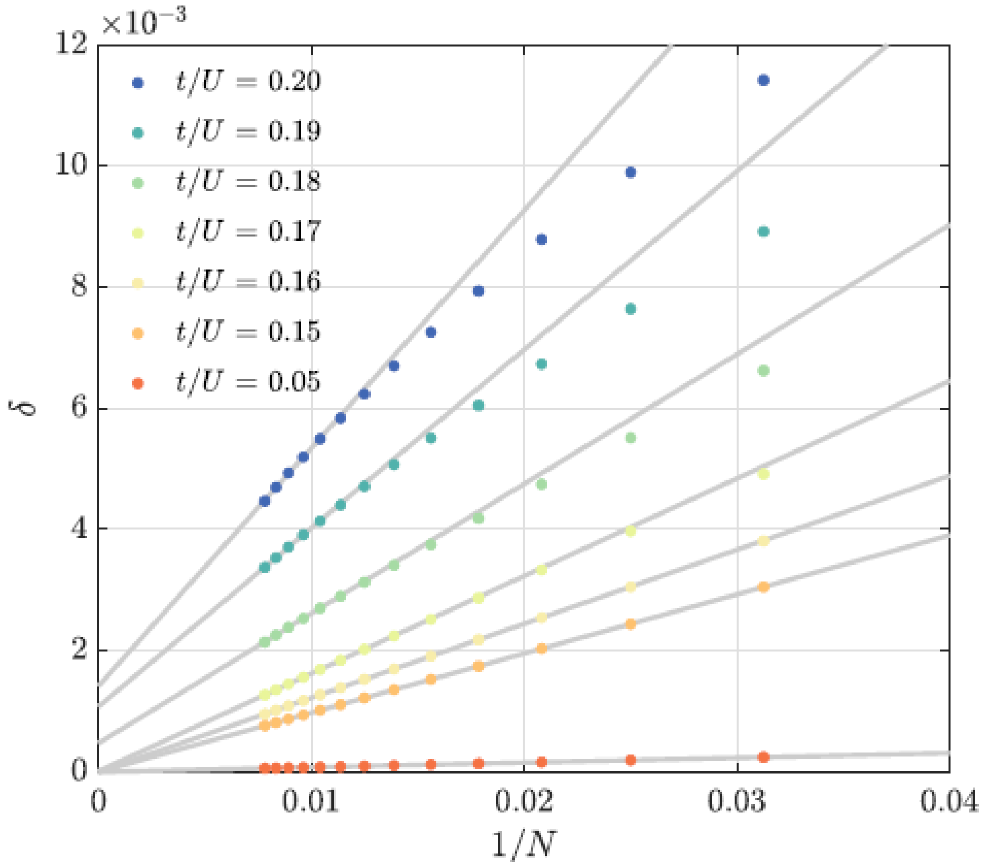
$$|\psi(x_1 + 1, \cdots, x_M + 1)| = \epsilon^{M/N} |\psi(x_1, \cdots, x_M)|$$

Notice this relation is independent of the interaction term U.

Suppression of NHSE in thermodynamic limit



$$\delta = \frac{1}{N} \sum_{j=1}^{N} |\bar{n}_j - \rho|$$



$$t_c/U = 0.17155.$$

Summary:

The exact solution of the two-particle quantum system with mass ratio 1/3 or 3 for arbitrary interaction strength can be obtained by extending the Bethe Ansatz method.

We exactly solve the 1D Bose-Hubbard model with unidirectional hopping and demonstrate the existence of superfluid-Mott transition at integer filling and the suppression of skin effect in Mott phase.

Thanks!